OCR A Level Maths A

Tues June 21st 2022

Pure Math and Mechanics

1. $|2x - 3| = 9$

squaring both sides

$4x^2 - 12x + 9 = 81$

$4x^2 - 12x = 72$

$4(x^2 - 3x - 18) = 0$

$4(x-6)(x+3) = 0$

$x = 6$ or $x = 3$

2a) $y = x^3$ to $y = x^3 - 8$

translation $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$

(or -8 units in y-direction)

b) $f(x) = x^3 - 8$

$y = x^3 - 8$

Make x the subject

$y + 8 = x^3$

$\sqrt[3]{y+8} = x$

$f^{-1}(x) = \sqrt[3]{x+8}$

c) Relation between $y = f(x)$ and $y = f^{-1}(x)$

They are reflections of each other in the line $y = x$.

3. $P(2, -5)$ PQ is diameter
$Q(3, 1)$

Centre is $\left(\frac{2+3}{2}, \frac{-5+1}{2}\right)$

C is $\left(\frac{5}{2}, -2\right)$

Length of radius is r

$$r = \sqrt{\left(3-\frac{5}{2}\right)^2 + (1--2)^2} = \frac{\sqrt{37}}{2}$$

Equation of circle

$(x-a)^2 + (y-b^2)^2 = r^2$

$\left(x-\frac{5}{2}\right)^2 + (y+2)^2 = \frac{37}{4}$

$x^2 - 5x + \frac{25}{4} + y^2 + 4y + 4 = \frac{37}{4}$

$x^2 + y^2 - 5x + 4y + 1 = 0$

4. 1st 2nd 3rd Arithmetic

a_a a_a+d a_a+2d

x y z

a_a a_a-4 a_a-8

$d = -4$

1st 2nd 3rd Geometric

x $\frac{15}{y}$ z

a_g $a_g \times r$ $a_g \times r^2$

Both "1st terms" $= x$

$\therefore$ $a_a = a_g = a$ (to simplify)

1st 2nd 3rd

Geometric Prog. $x, \frac{15}{y}, z$

$$\therefore \frac{\frac{15}{y}}{x} = \frac{z}{\frac{15}{y}} = r$$

$$\frac{15}{xy} = \frac{zy}{15} = r$$

$$\Rightarrow \frac{225}{y^2} = xz \quad (1)$$

Arithmetic prog.

$$y - x = z - y = -4$$

$$\therefore x = y + 4 \quad (3)$$

$$z = y - 4 \quad (4)$$

Using (1) $225 = xy^2z$

Using (3) and (4) in (1)

$$y^2(y+4)(y-4) = 225$$

$$y^2(y^2 - 16) = 225$$

$$y^4 - 16y^2 - 225 = 0$$

(as required)

b) solving quartic using Polynomial solver on calc

$y = 5, -5$

Question says positive integers $\therefore y = 5$

So $y = a - 4$

$5 = a - 4$

$a = 9 \quad \therefore x = 9$

$$\frac{15}{xy} = r$$

$$\frac{15}{9 \times 5} = \frac{1}{3} = r$$

Sum to $\infty = \frac{a}{1-r}$

$S_\infty = \frac{9}{1-\frac{1}{3}} = \frac{27}{2}$

5a) $y = \frac{2x-3}{4x^2+1}$ $\leftarrow u$ $\leftarrow v$

Quotient rule

$u = 2x-3$ $\quad v = 4x^2+1$
$u' = 2$ $\quad v' = 8x$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$2 = \frac{2(4x^2+1) - 8x(2x-3)}{(4x^2+1)^2}$

$2(4x^2+1)^2 = 8x^2+2-16x^2+24x$

$2(16x^4+8x^2+1)+8x^2-24x-2=0$

$32x^4+16x^2+2+8x^2-24x-2=0$

$32x^4+24x^2-24x=0$

$8x(4x^3+3x-3)=0$

$\therefore$ either $8x=0$

or $4x^3+3x-3=0$

(as required)

b) $f(0\cdot5) = 4\times0\cdot5^3+3\times0\cdot5-3 = -1$

$f(1) = 4\times1^3+3-3 = 4$

As there is a change of sign between f(0.5) and f(1) x coordinate lies between 0·5 and 1

c) $h(x) = 1 - \frac{4x^3}{3}$

$h'(x) = \frac{-12x^2}{3} = -4x^2$

as root lies between 0.5 and 1

$-4 \times 0.5^2 = -1$

$-4 \times 1 = -4$

$h'(x) \leqslant -1$

so iterative formula cannot cannot converge to x-coordinate of P.

d) Newton-Raphson

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ initial $x_0 = 0.5$

$= 0.5$ (puts 0.5 in ANS)

$f(x) = 4x^3 + 3x - 3$

$f'(x) = 12x^2 + 3$

$x_1 = ANS - \frac{4 \times ANS^3 + 3 \times ANS - 3}{12 \times ANS^2 + 3}$

$x_1 = \frac{2}{3}$

$x_2 = 0.644444$

$x_3 = 0.643955$

$x_4 = 0.643954$

$x_5 = 0.643954$

$\therefore x = 0.64395$ to 5dp

$y = \frac{2 \times 0.643954 - 3}{4 \times 0.643954^2 + 1}$

$y = -0.64396$ to 5dp

$P(0.64395, -0.64396)$

6. $y = \sqrt{2x+9}$ $\quad y = 0$

$0 = \sqrt{2x+9}$

$x = -4.5$

Meets x-axis at -4.5

$$\int_{-4.5}^{0} (2x+9)^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3} \times \frac{1}{2} \times (2x+9)^{\frac{3}{2}}$$

$$= \left[\frac{1}{3}(2x+9)^{\frac{3}{2}}\right]_{-4.5}^{0}$$

$$= \left(\frac{1}{3}(2\times 0+9)^{\frac{3}{2}}\right) - \left(\frac{1}{3}(2\times -4.5+9)^{\frac{3}{2}}\right)$$

$= 9$ ① $\quad = 0$

$y = 4e^{-2x} - 1$

$0 = 4e^{-2x} - 1$

$4e^{-2x} = 1$

$e^{-2x} = \frac{1}{4}$

$\ln e^{-2x} = \ln\left(\frac{1}{4}\right)$

$-2x = \ln\left(\frac{1}{4}\right)$

$x = \frac{\ln\left(\frac{1}{4}\right)}{-2} = \frac{-\ln 4}{-2}$

$= \frac{1}{2}\ln 4 = \frac{1}{2}\ln 2^2$

$= \ln 2$

$$\int_{0}^{\ln 2} (4e^{-2x} - 1)\,dx$$

$$= \left[-\frac{1}{2}\times 4 \times e^{-2x} - x\right]_{0}^{\ln 2}$$

$$= \left[-2e^{-2x} - x\right]_{0}^{\ln 2}$$

$$= \left(-2e^{-2\times\ln 2} - \ln 2\right) - \left(2e^{-0} - 0\right)$$

$$= -2e^{\ln 2^{-2}} - \ln 2 + 2$$
$$= -2 \times 2^{-2} - \ln 2 + 2$$
$$= -\frac{1}{2} - \ln 2 + 2$$
$$= +\frac{3}{2} - \ln 2 \quad (2)$$

Total area $= (1) + (2)$

$$= 9 + \frac{3}{2} - \ln 2$$
$$= \frac{21}{2} - \ln 2$$

7a) $m\sec\theta + 3\cos\theta = 4\sin\theta$

$$m\sec\theta + \frac{3}{\sec\theta} = 4\sin\theta$$

$\sec\theta = \frac{1}{\cos\theta}$

$$\frac{m\sec^2\theta + 3}{\sec\theta} = 4\sin\theta$$

$m\sec^2\theta + 3 = 4\sin\theta\sec\theta$ $\quad$ $\sec^2\theta = 1 + \tan^2\theta$

$$m(1 + \tan^2\theta) + 3 = \frac{4\sin\theta}{\cos\theta}$$

$$m + m\tan^2\theta + 3 = 4\tan\theta$$

$$m\tan^2\theta - 4\tan\theta + (m+3) = 0$$
(as required)

b) $0 < \theta < \pi$ $\quad$ m negative

Solve $m\tan^2\theta - 4\tan\theta + (m+3) = 0$
using quadratic formula
$a = m$, $b = -4$ $\quad$ $c = (m+3)$

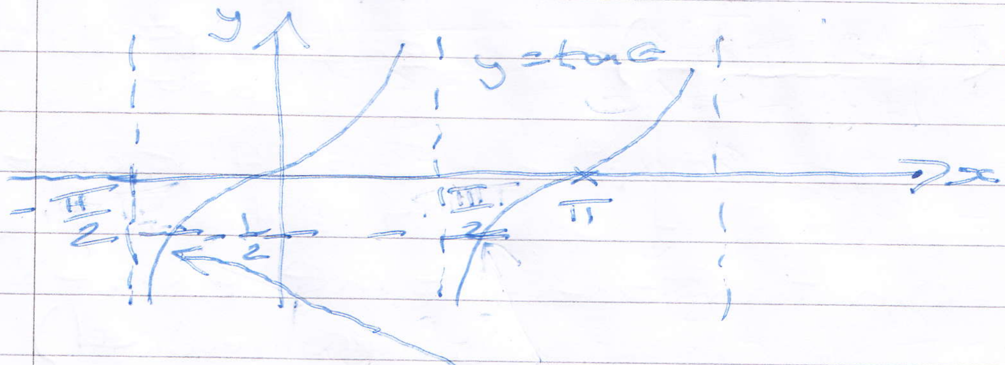
$$\tan\theta = --4 \pm \sqrt{(-4)^2 - 4m(m+3)}$$

As quadratic only has one solution for θ in interval
$b^2 - 4ac = 0$ (only one root)

$(-4)^2 - 4m(m+3) = 0$
$16 - 4m^2 - 12m = 0$
$4m^2 + 12m - 16 = 0$
$4(m^2 + 3m - 4) = 0$
$4(m+4)(m-1) = 0$
$m = -4$ or $m = 1$

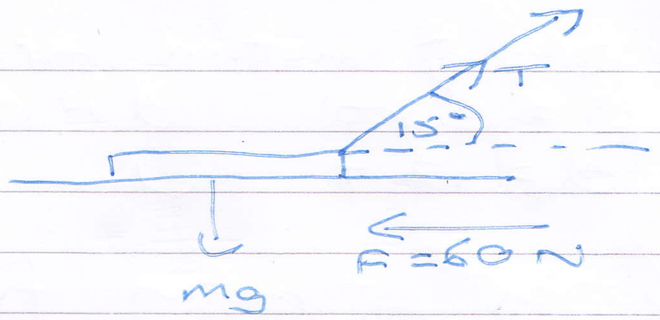
From question m is negative
$\therefore m = -4$

solve $-4\tan^2\theta - 4\tan\theta + (-4+3) = 0$
$-4\tan^2\theta - 4\tan\theta - 1 = 0$
$4\tan^2\theta + 4\tan\theta + 1 = 0$
$(2\tan\theta + 1)(2\tan\theta + 1) = 0$
$\tan\theta = -\frac{1}{2}$ $\quad \tan\theta = -\frac{1}{2}$



$\theta = -0.463647...$
$\therefore \theta = -0.463647... + \pi$
$= 2.6779...$
$= 2.68$ (3 sf)

8.



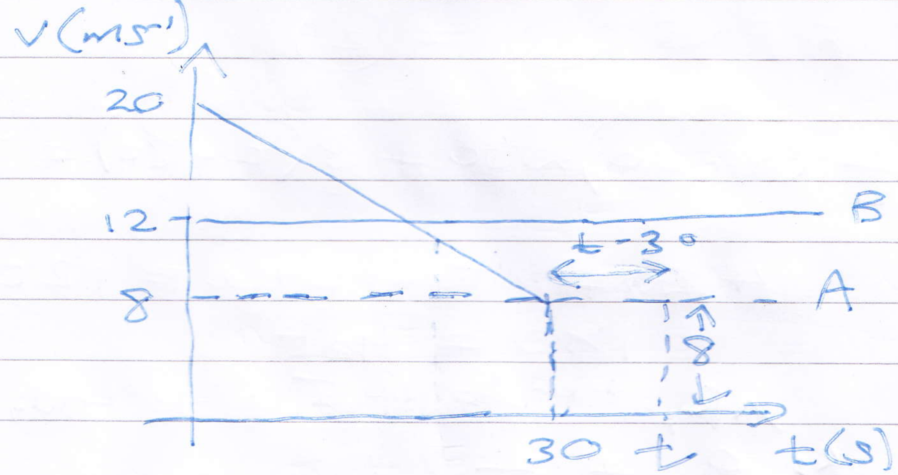
R (→)

$$T\cos 15^\circ - 60 = 0$$

$$T = \frac{60}{\cos 15^\circ} = 62.11657$$

$$= 62.1\,N \text{ (3sf)}$$

9.



a) s

$u = 20\,ms^{-1}$

$v = 8\,ms^{-1}$

$a = ?$

$t = 30$

$$v = u + at$$

$$a = \frac{v-u}{t}$$

$$a = \frac{8-20}{30} = -0.4\,ms^{-2}$$

b) In 30 seconds B travels

$$s = \frac{d}{t} \quad d = s \times t$$

$$d = 12 \times 30 = 360m$$

In 30 seconds A travels

(area of trapezium above)

$$\frac{1}{2}(20+8) \times 30 = 420\,m$$

Equation of motion for B

$d = s \times t$ d is distance travelled

$d = 12t$ (1)

Equation of motion for A

dist travelled

$= \frac{1}{2}(8+20)\times 30 + 8(t-30)$ (2)

For time to overtake set (1) = (2)

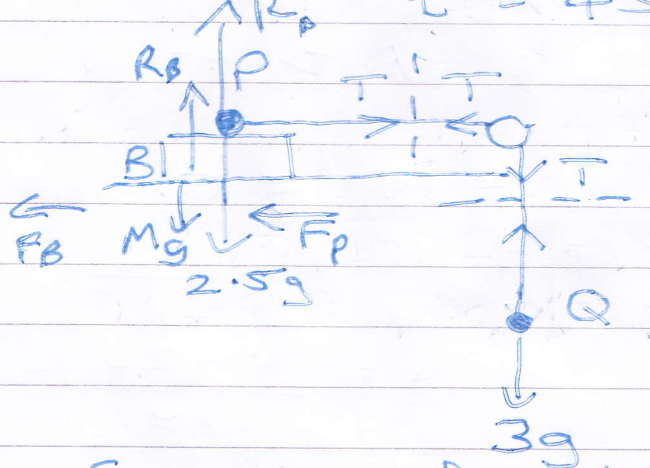
$\frac{1}{2}(8+20)\times 30 + 8(t-30) = 12t$

$420 + 8t - 240 = 12t$

$180 = 4t$

$t = 45$ seconds

10a)



$T = 16.8\,N$

Equation of motion for Q (↓)

$3 \times a = 3g - T$

$3a = 3\times 9.8 - 16.8$

$a = \frac{3\times 9.8 - 16.8}{3}$

$a = 4.2\ ms^{-2}$

b) Equation of motion for P ($\longrightarrow$)

$$2.5 \times 4.2 = T - F_p$$
$$2.5 \times 4.2 = 16.8 - F_p$$
$$F_p = 16.8 - 2.5 \times 4.2$$
$$F_p = 6.3 \text{ N}$$

Limiting friction $F = \mu R$

$$6.3 = \mu \times R_p \quad (1)$$

R ($\uparrow$) for P

$$R_p = 2.5g = 24.5 \text{N}$$

in (1) $\mu = \dfrac{6.3}{24.5} = 0.257$ (3sf)

c) ($\uparrow$) $R_B = Mg + 2.5g$ (2) where M is mass of B

$$\mu_B = \frac{5}{49}$$

$$F_B = \mu \times R_B$$

As B remains in equilibrium $F_p = F_B$
$= 6.3\text{N}$

$$6.3 = \frac{5}{49} \times R_B$$

$$R_B = \frac{49 \times 6.3}{5} = 61.74\text{N}$$

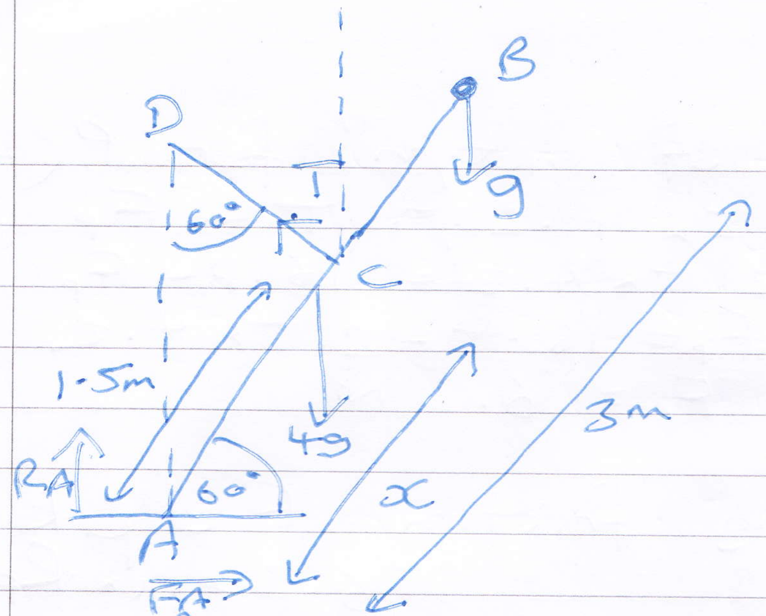
Using (2)

$$61.74 = M \times 9.8 + 2.5 \times 9.8$$

$$M = \frac{61.74 - 2.5 \times 9.8}{9.8}$$

$$M = 3.8 \text{ kg}$$

11.



$AC = x$ cm $\quad AB = 3m$

a) $M(A)$

$x \times T = 3\cos 60° \times g + 1.5\cos 60° \times 4g$

$x \times T = 3 \times \frac{1}{2} \times g + 1.5 \times \frac{1}{2} \times 4g$

$xT = \frac{9}{2} g$

$T = \frac{9g}{2x}$ N

b) $\mu = \frac{9\sqrt{3}}{35}$

$R(\uparrow)$ $T\cos 60° + R_A = 4g + g$

$\frac{9g}{2x} \times \frac{1}{2} + R_A = 5g$

$R_A = 5g - \frac{9g}{4x}$

Limiting friction

$F_A = \mu R_A$

$R(\rightarrow)$

$F_A = T\sin 60°$

$F_A = \frac{9g}{2x} \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}g}{4x}$

$$\therefore \frac{9\sqrt{3}\,g}{4x} = \frac{9\sqrt{3}}{35} \times \left(5g - \frac{9g}{4x}\right)$$

$$\frac{\cancel{9}\cancel{\sqrt{3}} \times 35 \times g}{\cancel{9} \times 4 \times \cancel{\sqrt{3}} \times x} = 5g - \frac{9g}{4x}$$

$$\frac{35g}{4x} = 5g - \frac{9g}{4x}$$

$$\frac{35g}{4x} + \frac{9g}{4x} = 5g$$

$$\frac{44\cancel{g}}{4x} = 5\cancel{g}$$

$$\frac{11}{x} = 5$$

$$x = \frac{11}{5} = 2.2\text{ m}$$

12. $\underset{\sim}{v} = (1-2t)\underset{\sim}{i} + (2t^2 + t - 13)\underset{\sim}{j}$

a) If stationary $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of v would be zero

$\underset{\sim}{i}$ components $1-2t=0$

$2t=1$

$t=\frac{1}{2}$

$\underset{\sim}{j}$ components $2t^2+t-13=0$

Using "polynomial solver"

$t = 2.3117 \quad t = -2.8117$

No value of t is common to both components, so P is never stationary.

b) $\underset{\sim}{v} = (1-2t)\underset{\sim}{i} + (2t^2+t-13)\underset{\sim}{j}$

differentiating

$\underset{\sim}{a} = -2\underset{\sim}{i} + (4t+1)\underset{\sim}{j} \quad ms^{-2}$

or $\underset{\sim}{a} = \begin{pmatrix} -2 \\ 4t+1 \end{pmatrix}$

c) $\underset{\sim}{F} = m\underset{\sim}{a} = 0.5\begin{pmatrix} -2 \\ 4t+1 \end{pmatrix}$

$\underset{\sim}{F} = \begin{pmatrix} -1 \\ 2t+0.5 \end{pmatrix}$

$v = (1-2t)\underset{\sim}{i} + (2t^2+t-13)\underset{\sim}{j}$

when moving in direction

$-2\underset{\sim}{i} + \underset{\sim}{j}$

$$\frac{1-2t}{-2} = \frac{2t^2+t-13}{1}$$

$1-2t = -2(2t^2+t-13)$

$1-2t+4t^2+2t-26=0$

$4t^2-25=0$

$t=2.5$ or $t=-2.5$

d) To get $\underset{\sim}{s}$ we integrate $\underset{\sim}{v}$

$\underset{\sim}{v} = \begin{pmatrix} 1-2t \\ 2t^2+t-13 \end{pmatrix}$

Integrating

$$\underset{\sim}{s} = \begin{pmatrix} t-t^2 \\ \frac{2t^3}{3}+\frac{t^2}{2}-13t \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

at $t=1$ $\quad \underset{\sim}{s} = \begin{pmatrix} 0 \\ \frac{1}{6} \end{pmatrix}$

$$\begin{pmatrix} 0 \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 - 1^2 \\ \frac{2}{3} + \frac{1}{2} - 13 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0 + c_1 \\ -\frac{71}{6} + c_2 \end{pmatrix}$$

$\therefore \quad c_1 = 0$

$c_2 = \frac{1}{6} + \frac{71}{6} = \frac{72}{6} = 12$

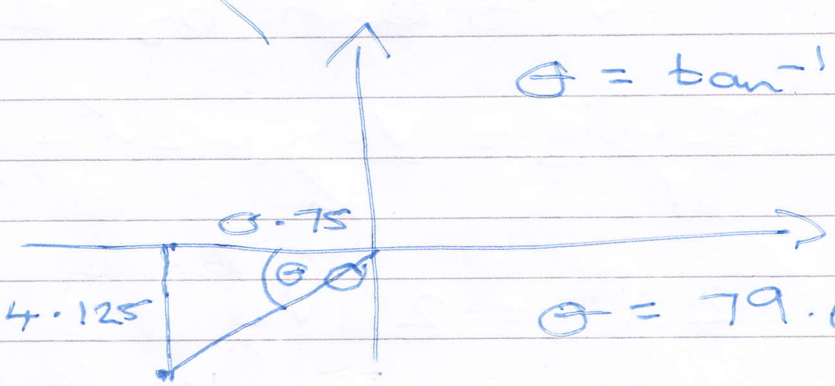
$$\therefore \quad \underset{\sim}{s} = \begin{pmatrix} t - t^2 \\ \frac{2}{3}t^3 + \frac{t^2}{2} - 13t + 12 \end{pmatrix}$$

at $t = 1.5$

$$\underset{\sim}{s} = \begin{pmatrix} 1.5 - 1.5^2 \\ \frac{2}{3} \times 1.5^3 + \frac{1.5^2}{2} - 13 \times 1.5 + 12 \end{pmatrix}$$

$$\underset{\sim}{s} = \begin{pmatrix} -0.75 \\ -4.125 \end{pmatrix}$$

$$\theta = \tan^{-1}\left(\frac{4.125}{0.75}\right)$$



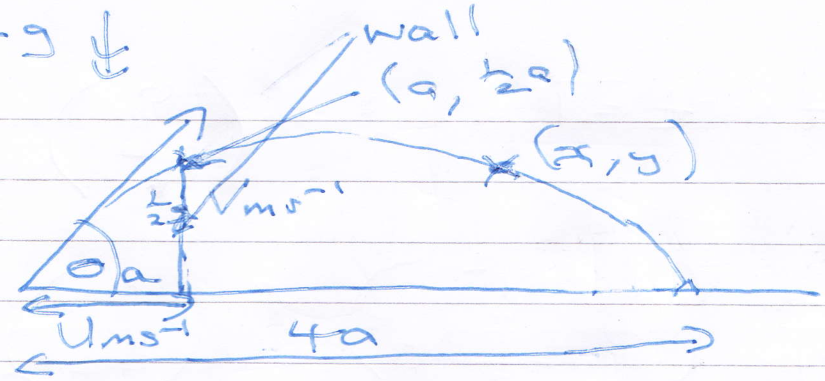
$\theta = 79.695^\circ$

Bearing $= 360 - 90 - 79.695^\circ$

$= 190.304^\circ = 190^\circ$

(to nearest degree)

13. $a = -g$ ↓



(→) motion constant velocity $U ms^{-1}$ to get to point (x, y)

$x = Ut$ (1)

(↑) motion

$s = y$

$u = V ms^{-1}$

$v =$

$a = -g$

$t = t$

$s = ut + \frac{1}{2}at^2$

$y = Vt + \frac{1}{2} \times -g \times t^2$

from (1) $t = \frac{x}{U}$

$y = V \times \frac{x}{U} - \frac{1}{2} g \times \frac{x^2}{U^2}$

$\times$ through by $2U^2$

$2U^2 y = \frac{V \times x}{\cancel{U}} \times 2U^{\cancel{2}} - \frac{1}{\cancel{2}} g \frac{x^2}{\cancel{U^2}} \times \cancel{2U^2}$

$2U^2 y = 2UVx - gx^2$ (3)

(as required)

b) at point where it clears wall

($\uparrow$) $s = \frac{1}{2}a$ passes through $(a, \frac{1}{2}a)$

$u = V$

$v =$

$a = -g$

t

($\rightarrow$) motion $a = Ut$

from (3) $2U^2y = 2UVx - gx^2$

$x = a, \; y = \frac{1}{2}a$

$\cancel{2}^{1}U^2 \times \frac{a}{\cancel{2}_{1}} = 2UVa - ga^2$ (4)

$aU^2 = 2UVa - ga^2$

divide through by a

$U^2 = 2UV - ga$ (5)

B hits ground at $y=0, x=4a$

Using (3)

$2U^2 \times 0 = 2UV \times 4a - g(4a)^2$

$0 = 8aUV - 16a^2g$

$0 = 8a(UV - 2ag)$

$\therefore UV - 2ag = 0$ (6)

Solve (5) and (6) simultaneously

(6) gives $V = \frac{2ag}{U}$ sub in (5)

$U^2 = 2\cancel{U}^{1} \times \frac{2ag}{\cancel{U}_{1}} - ag$

$$u^2 = 3ag$$
$$u = \sqrt{3ag}$$

in (6)

$$v \times \sqrt{3ag} = 2ag$$

$$v = \frac{2ag}{\sqrt{3ag}}$$

$$\therefore \tan\theta = \frac{v}{u} = \frac{\frac{2ag}{\sqrt{3ag}}}{\sqrt{3ag}}$$

$$= \frac{2ag}{\sqrt{3ag}} \div \sqrt{3ag}$$

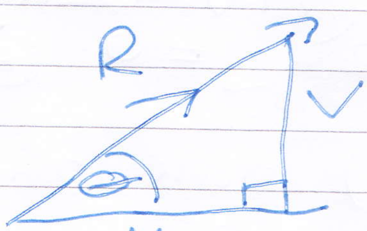
$$= \frac{2ag}{\sqrt{3ag}} \times \frac{1}{\sqrt{3ag}}$$

$$\tan\theta = \frac{2\cancel{a}\cancel{g}}{3\cancel{a}\cancel{g}} = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 33.690^\circ$$

$$= 33.7^\circ (3sf)$$

c)



$$R^2 = u^2 + v^2$$

$$R^2 = 3ag + \frac{4a^2g^2}{3ag}$$

$$R^2 = 3ag + \frac{4ag}{3}$$

Given $R = 54.6 \, ms^{-1}$

$g = 9.8 \, ms^{-2}$

$$54.6^2 = \frac{13}{3} \times 9.8^2 \times a$$

$$a = \frac{54.6^2 \times 3}{13 \times 9.8^2} = 70.2$$

$$a = 70.2 \, m$$

d)

$s = ?$

$u = V = \frac{2ag}{\sqrt{3ag}} = \frac{2 \times 70.2 \times 9.8}{\sqrt{3 \times 70.2 \times 9.8}}$

$v = 0$ $= 30.28663071$

$a = -g$

t

R (↑)

$$v^2 = u^2 + 2as$$

$$\frac{v^2 - u^2}{2a} = s$$

$$\frac{0^2 - 30.28663071^2}{2 \times -9.8}$$

$$= 46.7999\ldots \, m$$

$$= 46.8 \, m \quad (3sf)$$

e) taking into account

- size of B
- wind / weather
- thickness of wall
- clearance of ball at top of wall