OCR
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YORKSHIRE MATHS TUTOR

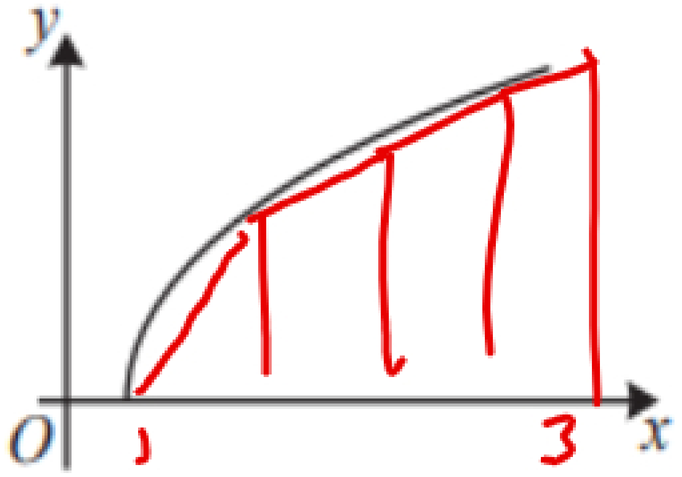
Tuesday 7 June 2022 – Afternoon

A Level Mathematics A

H240/01 Pure Mathematics

Time allowed: 2 hours

1



The diagram shows part of the curve $y = \sqrt{x^2 - 1}$.

(a) Use the trapezium rule with 4 intervals to find an estimate for $\int_1^3 \sqrt{x^2 - 1}\,dx$.

Give your answer correct to **3** significant figures. **[4]**

a)

$h = 0.5$

x	1	1.5	2	2.5	3
$y = \sqrt{x^2 - 1}$	0	1.12	1.73	2.29	2.83

$$\text{area} = \frac{1}{2} \times 0.5 \times \left[0 + 2.83 + 2(1.12 + 1.73 + 2.29) \right]$$

$$= 3.2775$$

$$= 3.28 \text{ (3 sf)}$$

(b) State whether the value from part **(a)** is an under-estimate or an over-estimate, giving a reason for your answer. **[1]**

(c) Explain how the trapezium rule could be used to obtain a more accurate estimate. **[1]**

b) From diagram, trapezia below curve so underestimate

c) Make a smaller width (< 0.5) to have more trapezia

2 **(a)** Given that a and b are real numbers, find a counterexample to disprove the statement that, if $a > b$, then $a^2 > b^2$. **[1]**

if $a = 1$, $b = -1$

$a > b$ as $1 > -1$

but for $a^2 > b^2$

$1 = 1$ not $1 > 1$

so this is a contradiction

(b) A student writes the statement that $\sin x° = 0.5 \Longleftrightarrow x° = 30°$.

(i) Explain why this statement is incorrect. **[1]**

x could also be $150°$, $330°$ etc, not just $30°$

(ii) Write a corrected version of this statement. **[1]**

$\sin x° \Longleftarrow x° = 30°$

$x = 30° + 360n°$ (n integer)

or

$x = 150° + 360n°$

(c) Prove that the sum of four consecutive multiples of 4 is always a multiple of 8. [3]

Let multiple of 4 be $4n$

$\therefore$ next is $4n+4$

$4n+8$

$4n+12$

Sum $= 16n+24$

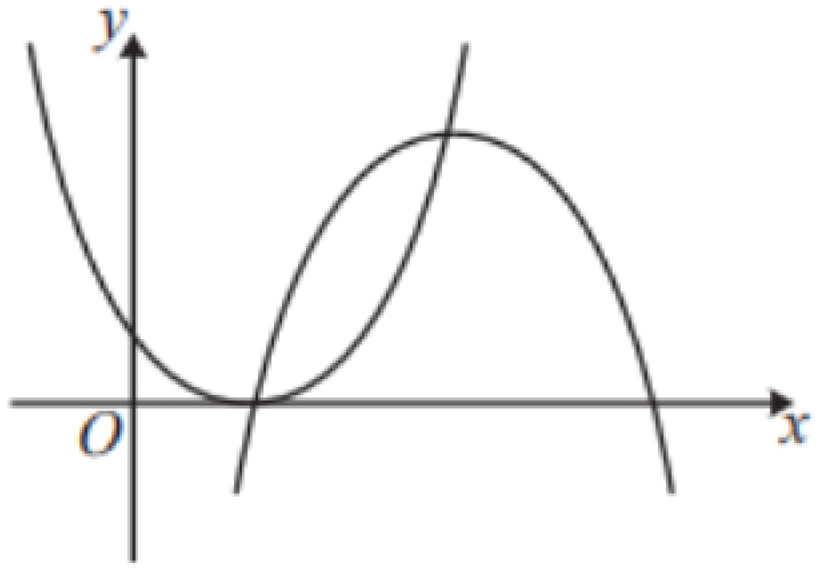
$= 8(2n+3)$

which is a multiple of 8

3 (a) In this question you must show detailed reasoning.

Find the coordinates of the points of intersection of the curves with equations $y = x^2 - 2x + 1$ and $y = -x^2 + 6x - 5$. **[4]**

(b) The diagram shows the curves $y = x^2 - 2x + 1$ and $y = -x^2 + 6x - 5$.
This diagram is repeated in the Printed Answer Booklet.



a) $x^2 - 2x + 1 = -x^2 + 6x - 5$

$2x^2 - 8x + 6 = 0$

$2(x^2 - 4x + 3) = 0$

$2(x - 3)(x - 1) = 0$

$x = 3 \qquad x = 1$

$y = 9 - 6 + 1 \qquad y = 1 - 2 + 1$

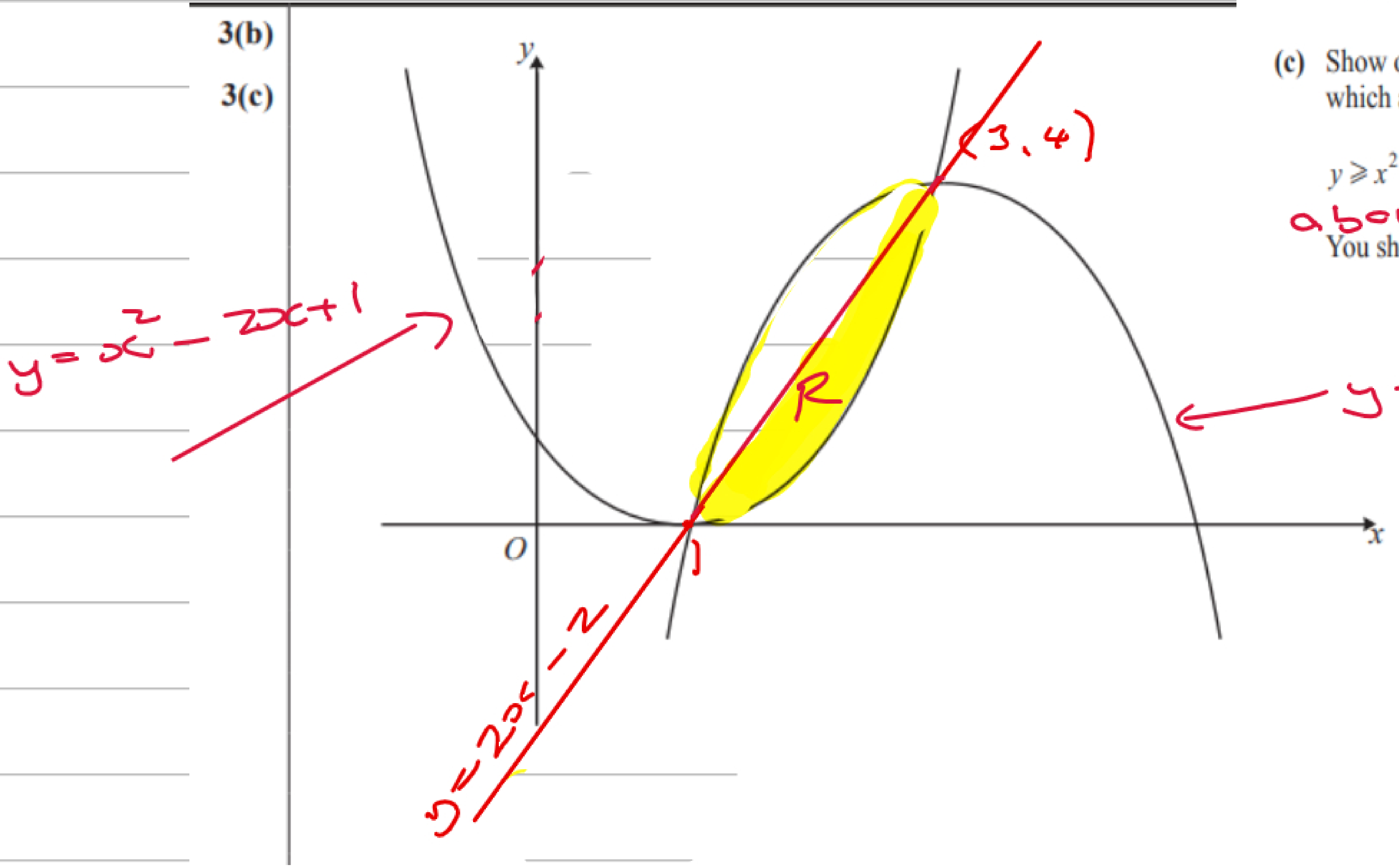
$y = 4 \qquad y = 0$

$(3, 4)$ and $(1, 0)$

(b) The diagram shows the curves $y = x^2 - 2x + 1$ and $y = -x^2 + 6x - 5$.
This diagram is repeated in the Printed Answer Booklet.

x	0	1	2	3
$y = 2x - 2$	-2	0	2	4

On the diagram in the Printed Answer Booklet, draw the line $y = 2x - 2$. **[2]**



(c) Show on your diagram in the Printed Answer Booklet the region of the x-y plane within which all three of the following inequalities are satisfied.

$y \geqslant x^2 - 2x + 1$ $\qquad$ $y \leqslant -x^2 + 6x - 5$ $\qquad$ $y \leqslant 2x - 2$

above $\qquad$ below $\qquad$ below

You should indicate the region for which all the inequalities hold by labelling the region R. **[1]**

$y = -x^2 - 2x + 1$

4 (a) Write $2x^2 + 6x + 7$ in the form $p(x+q)^2 + r$, where p, q and r are constants. [3]

(b) State the coordinates of the minimum point on the graph of $y = 2x^2 + 6x + 7$. [2]

a) $2\left(x^2 + 3x + \frac{7}{2}\right)$

$= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{7}{2}\right]$

$= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{14}{4}\right]$

$= 2\left[\left(x + \frac{3}{2}\right)^2 + \frac{5}{4}\right]$

$= 2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$

b) $x = -\frac{3}{2}$, $y = \frac{5}{2}$

$\left(-\frac{3}{2}, \frac{5}{2}\right)$

(c) Hence deduce

- the minimum value of $2\tan^2\theta + 6\tan\theta + 7$,
- the smallest positive value of θ, in degrees, for which the minimum value occurs. **[3]**

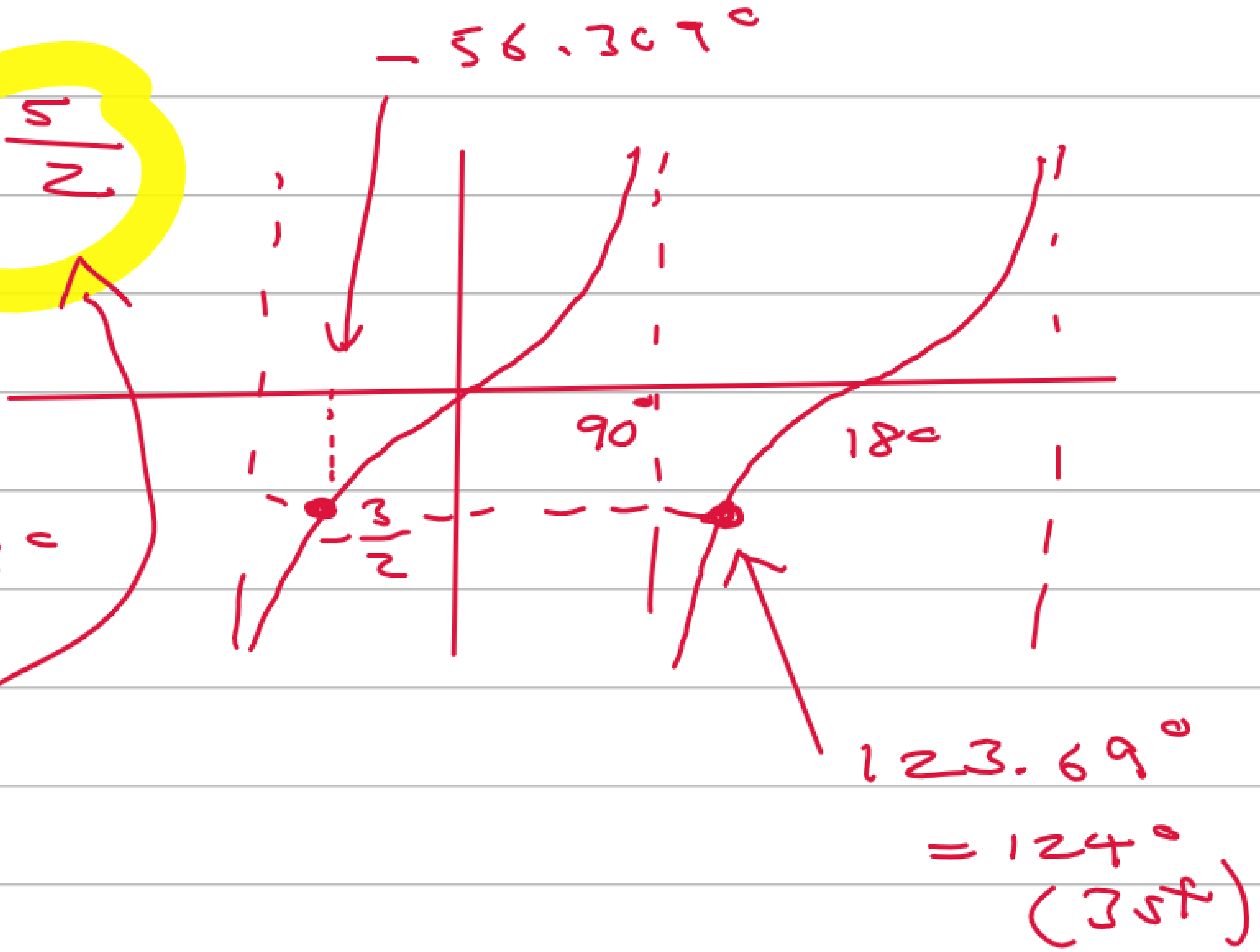
$$2x^2 + 6x + 7 = 2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$$

$$2\left(\tan x + \frac{3}{2}\right)^2 + \frac{5}{2}$$

$$\tan x = -\frac{3}{2}$$

$$x = -56.309^\circ$$

$$180 - 56.309 = 123.69^\circ$$



Min value = $\frac{5}{2}$

when $\theta = 124^\circ$ (3sf)

5 (a) The graph of $y = 2^x$ can be transformed to the graph of $y = 2^{x+4}$ **either** by a translation **or** by a stretch.

(i) Give full details of the translation. 4 unit left **[2]** translate $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$

(ii) Give full details of the stretch. **[2]**

(ii) $y = 2^{x+4} = 2^x \times 2^4$

$= 16 \times 2^x$

in y-direction scale factor 16

(b) In this question you must show detailed reasoning.

Solve the equation $\log_2(8x) = 1 - \log_2(1-x)$. **[4]**

$$\log_2(8x) = 1 - \log_2(1-x)$$

$$\log_2(8x) + \log_2(1-x) = 1$$

$$\log_2(8x(1-x)) = 1$$

$$\log_2(8x - 8x^2) = 1$$

$$8x - 8x^2 = 2^1$$

$$8x - 8x^2 = 2$$

$$0 = 8x^2 - 8x + 2$$

$$0 = 2(4x^2 - 4x + 1)$$

$$0 = 2(2x-1)(2x-1)$$

$x = \frac{1}{2}$ or $x = \frac{1}{2}$

$x = \frac{1}{2}$

6 (a) Find the first four terms in the expansion of $(3+2x)^5$ in ascending powers of x. [4]

$${}^5C_0\, 3^5 (2x)^0 + {}^5C_1\, 3^4 (2x)^1 + {}^5C_2\, 3^3 (2x)^2 + {}^5C_3\, 3^2 (2x)^3$$

$$= 243 + 810x + 1080x^2 + 720x^3 + \ldots$$

(b) Hence determine the coefficient of y^3 in the expansion of $(3+2y+4y^2)^5$.

$(3+2x)^5$

$2x = 2y + 4y^2$

$x = y + 2y^2$

$$= 243 + 810(y+2y^2) + 1080(y+2y^2)^2 + 720(y+2y^2)^3$$

$$= 243 + 810y + 1620y^2 + 1080(y^2 + 4y^3 + 4y^4) + 720y^3 + \ldots$$

y^3 terms are $1080 \times 4y^3 + 720y^3 = 5040y^3$

Coefficient $= 5040$

7 A curve has equation $2x^3 + 6xy - 3y^2 = 2$. ① implicit differentiation

Show that there are no points on this curve where the tangent is parallel to $y = x$ **[8]**

gradient = 1

$$6x^2 + 6x\frac{dy}{dx} + 6y - 6y\frac{dy}{dx} = 0$$

if $\frac{dy}{dx} = 1$

$$6x^2 + 6x + 6y - 6y = 0$$

$$6x^2 + 6x = 0$$

$$6x(x+1) = 0$$

$$x = 0 \quad \text{or} \quad x = -1$$

sub $x = 0$ in ① $2 \times 0^3 + 6 \times 0 \times -3 \times y^2 = 2$

$$-3y^2 = 2$$

impossible as y^2 either 0 or +ve

BitPaper Share Knowledge

$x = -1$

$$2\times(-1)^3 + 6\times(-1)y - 3y^2 = 2$$

$$-2 - 6y - 3y^2 = 2$$

$$3y^2 + 6x + 4 = 0$$

$$b^2 - 4ac = \sqrt{6^2 - 4\times3\times4} = \sqrt{-12}$$

as $b^2 - 4ac < 0$ no real roots

So tangent never parallel to $y = x$

8 **(a)** Substance A is decaying exponentially such that its mass is m grams at time t minutes. Find the missing values of m and t in the following table.

t	0	10	20	50
m	1250	750	450	97.2

[2]

(b) Substance B is also decaying exponentially, according to the model $m = 160e^{-0.055t}$, where m grams is its mass after t minutes.

a) $m = ke^{-at}$

$t = 0, \quad m = 1250$

$1250 = k \times e^{0}$

$k = 1250$

$t = 10, \quad m = 750$

$750 = 1250 \times e^{-a \times 10}$

$e^{-10a} = \frac{750}{1250}$

$$e^{-10a} = \frac{750}{1250}$$

$$-10a = \ln\left(\frac{750}{1250}\right)$$

$$a = \frac{\ln\left(\frac{750}{1250}\right)}{-10}$$

$$a = 0.05108$$

at $t = 50$, $m = 1250 \times e^{-0.05108 \times 50} = 97.2\text{g}$

when $m = 450$ $\quad 450 = 1250 \times e^{-0.05108 \times t}$

$$\frac{450}{1250} = e^{-0.05108\,t}$$

$$\ln\left(\frac{450}{1250}\right) = -0.05108 \times t$$

$$t = \frac{\ln\left(\frac{450}{1250}\right)}{-0.05108} \leftarrow 20.001 \text{ minutes}$$

(b) Substance B is also decaying exponentially, according to the model $m = 160e^{-0.055t}$, where m grams is its mass after t minutes.

(i) Determine the value of t for which the mass of substance B is half of its original mass. **[3]**

(ii) Determine the rate of decay of substance B when $t = 15$. **[3]**

(i) $m = 160e^{-0.055t}$

at $t=0$, $m = 160$

$80 = 160e^{-0.055t}$

$\frac{80}{160} = e^{-0.055t}$

$\ln\left(\frac{1}{2}\right) = -0.055t$

$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.055} = 12.6$ minutes (3sf)

(ii) $m = 160 e^{-0.055t}$

$\frac{dm}{dt} = 160 \times -0.055 e^{-0.055t}$

at $t = 15$

$\frac{dm}{dt} = 160 \times -0.055 \times e^{-0.055 \times 15}$

$= -3.86$ g/minute (3 sf)

c) for A $m = 1250 \times e^{-0.05108 \times t}$

$\frac{dm}{dt} = 1250 \times -0.05108 \times e^{-0.05108t}$

at $t = 15$

$\frac{dm}{dt} = -29.7$ g/minute

$\therefore$ A decaying at a faster rate

9 Use the substitution $x = 2\sin\theta$ to show that $\int_1^{\sqrt{3}} \sqrt{4-x^2}\,dx = \frac{1}{3}\pi$. **[7]**

$x = 2\sin\theta$ limits

$\frac{dx}{d\theta} = 2\cos\theta$ $x = 1$ $\frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}$

$x = \sqrt{3}$ $\frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

$dx = 2\cos\theta\,d\theta$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{4(1-\sin^2\theta)} \times 2\cos\theta\,d\theta$$

$= \cos^2\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sqrt{1-\sin^2\theta} \times 2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \underbrace{4\cos\theta\cos\theta}\, d\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$\therefore 4\cos^2\theta = 2\cos 2\theta + 2$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\cos 2\theta + 2 \quad d\theta$$

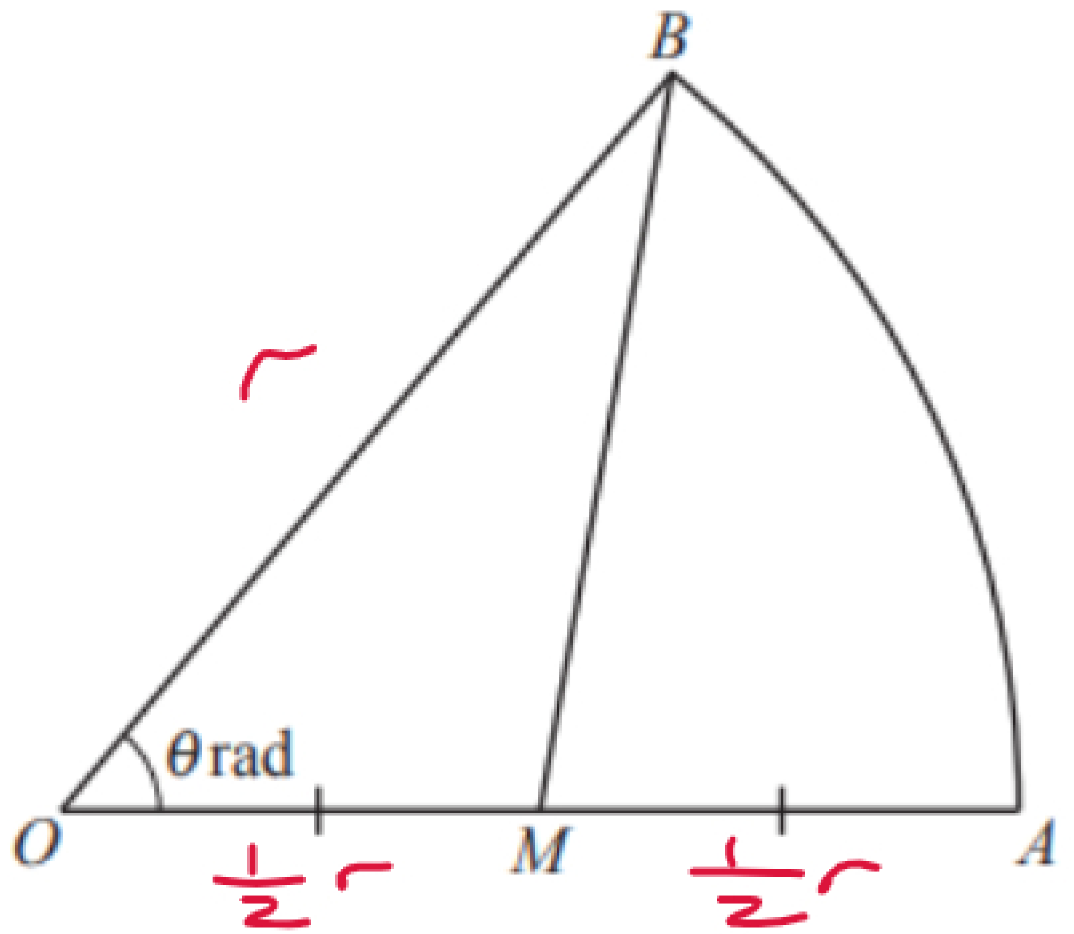
$$= \Big[\sin 2\theta + 2\theta \Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\sin\left(2\times\frac{\pi}{3}\right) + \frac{2\times\pi}{3} \right) - \left(\sin\left(\frac{2\pi}{6}\right) + \frac{2\pi}{6} \right)$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} \right) - \left(\frac{\sqrt{3}}{2} - \frac{\pi}{3} \right)$$

$$= \frac{\pi}{3} \quad (\text{as required})$$

10



The diagram shows a sector *OAB* of a circle with centre *O* and radius *OA*. The angle *AOB* is θ radians. *M* is the mid-point of *OA*. The ratio of areas *OMB* : *MAB* is 2:3.

(a) Show that $\theta = 1.25 \sin\theta$. **[4]**

a) area sector OBA $= \frac{1}{2}r^2\theta$

$\therefore$ area OMB $= \frac{2}{5} \times \frac{1}{2}r^2\theta$ (1)

from area $\triangle = \frac{1}{2}ab\sin C$

area OMB $=$ (2)

$\frac{1}{2} \times r \times \frac{1}{2}r \times \sin\theta$

set (1) = (2)

$\frac{1}{5}\cancel{r^2}\theta = \frac{1}{4}\cancel{r^2}\sin\theta$

$\theta = \frac{5}{4}\sin\theta$

$\theta = 1.25\sin\theta$ (as required)

The equation $\theta = 1.25\sin\theta$ has only one root for $\theta > 0$.

(b) This root can be found by using the iterative formula $\theta_{n+1} = 1.25\sin\theta_n$ with a starting value of $\theta_1 = 0.5$.

- Write down the values of θ_2, θ_3 and θ_4.
- Hence find the value of this root correct to **3** significant figures. **[3]**

$\theta_{n+1} = 1.25\sin\theta_n$

type 0.5 on calculator
1.25 × ANS

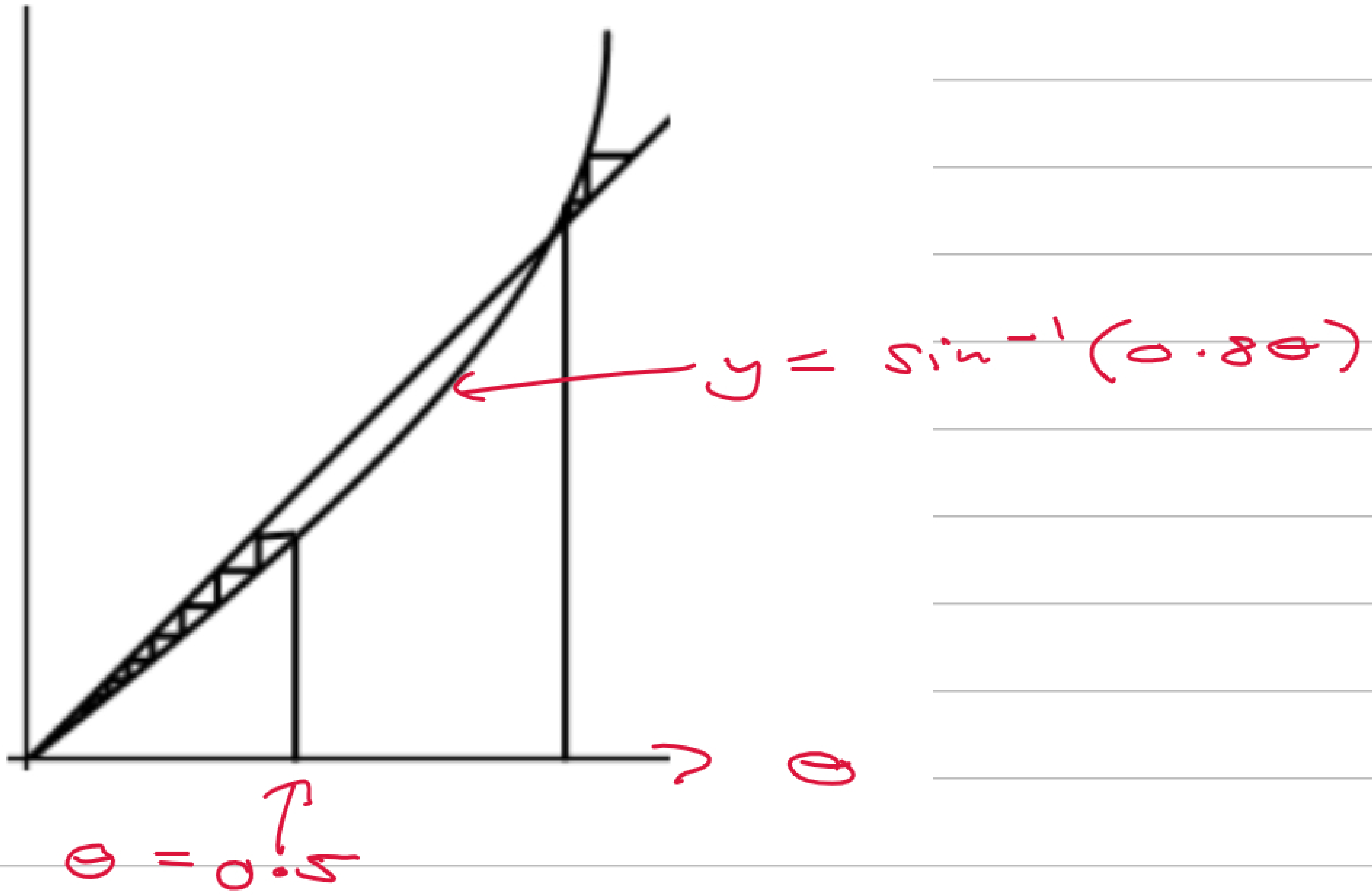
$\theta_2 = 0.599$
$\theta_3 = 0.705$
$\theta_4 = 0.810$

Keep pressing = on calculator, eventually converges on root = 1.13 (3 sf)

(c) The diagram in the Printed Answer Booklet shows the graph of $y = 1.25\sin\theta$, for $0 \leqslant \theta \leqslant \pi$.

- Use this diagram to show how the iterative process used in **(b)** converges to this root.
- State the type of convergence. **[3]**

(d) Draw a suitable diagram to show why using an iterative process with the formula $\theta_{n+1} = \sin^{-1}(0.8\theta_n)$ does not converge to the root found in **(b)**. **[2]**



11 The gradient function of a curve is given by $\frac{dy}{dx} = \frac{3x^2 \ln x}{e^{3y}}$.

The curve passes through the point (e, 1).

(a) Find the equation of this curve, giving your answer in the form $e^{3y} = f(x)$. **[6]**

integrate by parts

$$\int e^{3y}\,dy = \int 3x^2 \ln x \,dx$$

$u = \ln x$ $v = x^3$

$u' = \frac{1}{x}$ $v' = 3x^2$

$$\frac{1}{3}e^{3y} = x^3 \ln x - \int x^2 \,dx + c$$

$$\frac{1}{3}e^{3y} = x^3 \ln x - \frac{x^3}{3} + c$$

at $x = e$, $y = 1$

$$\frac{1}{3}e^3 = e^3 \times \underbrace{\ln e}_{=1} - \frac{e^3}{3} + c$$

$$\frac{2}{3}e^3 - e^3 = c$$

$$c = -\frac{1}{3}e^3$$

$$\frac{1}{3}e^{3y} = x^3 \ln x - \frac{x^3}{3} + c$$

$$\frac{1}{3}e^{3y} = x^3 \ln x - \frac{x^3}{3} - \frac{1}{3}e^3$$

× through by 3

$$e^{3y} = 3x^3 \ln x - x^3 - e^3$$

(b) Show that, when $x = e^2$, the y-coordinate of this curve can be written as $y = a + \frac{1}{3}\ln\left(be^3 + c\right)$, where a, b and c are constants to be determined. **[3]**

$$e^{3y} = 3(e^2)^3 \times \ln(e^2) - (e^2)^3 - e^3$$

$$e^{3y} = 3e^6 \times 2\ln e - e^6 - e^3$$

$$e^{3y} = 6e^6 - e^6 - e^3$$

$$e^{3y} = 5e^6 - e^3$$

$$e^{3y} = 5e^6 - e^3$$

$$\ln(a) + \ln(b) = \ln(ab)$$

log laws

$$\ln(e^{3y}) = \ln(5e^6 - e^3)$$

$$3y = \ln(e^3(5e^3 - 1))$$

$$3y = \ln(e^3) + \ln(5e^3 - 1)$$

$$3y = 3 + \ln(5e^3 - 1)$$

$$y = 1 + \frac{1}{3}(5e^3 - 1)$$

12 A curve has parametric equations $x = \frac{1}{t}$, $y = 2t$. The point P is $\left(\frac{1}{p}, 2p\right)$.

(a) Show that the equation of the tangent at P can be written as $y = -2p^2x + 4p$. **[4]**

$x = t^{-1}$ $\quad$ $y = 2t$

$\frac{dx}{dt} = -\frac{1}{t^2}$ $\quad$ $\frac{dy}{dt} = 2$

$\frac{dy}{dx} = \frac{2}{-\frac{1}{t^2}} = -2t^2$

when $y = 2t$
$2p = 2t$
$p = t$
$\therefore m = -2p^2$

$y - y_1 = m(x - x_1)$
$y - 2p = -2p^2\left(x - \frac{1}{p}\right)$
$y - 2p = -2p^2x + 2p$
$y = -2p^2x + 4p$ (as required)

The tangent to this curve at P crosses the x-axis at the point A and the normal to this curve at P crosses the x-axis at the point B.

$y = 0$

(b) Show that the ratio $PA : PB$ is $1 : 2p^2$. **[8]**

at A $\quad 0 = -2p^2x + 4p$

$0 = p(-2px + 4)$

either $p = 0$ or $-2px + 4 = 0$

$4 = 2px$

$\frac{2}{p} = x$ $\quad A\left(\frac{2}{p}, 0\right)$

gradient of normal at P $= \frac{1}{2p^2}$

$y - y_1 = m(x - x_1)$

$y - 2p = \frac{1}{2p^2}\left(x - \frac{1}{p}\right)$

$y - 2p = \frac{1}{2p^2}x - \frac{1}{2p^3}$

normal is $\quad y = \frac{1}{2p^2}x + 2p - \frac{1}{2p^3}$

at B $\quad y = 0$

$$0 = \frac{1}{2p^2}x + 2p - \frac{1}{2p^3}$$

$$\frac{x}{2p^2} = \frac{1}{2p^2} - 2p$$

$$x = \frac{2p^2}{2p^3} - 2p \times 2p^2$$

$$x = \frac{1}{p} - 4p^3$$

$B\left(\frac{1}{p} - 4p^3, 0\right)$

$A\left(\frac{2}{p}, 0\right)$

$P\left(\frac{1}{p}, 2p\right)$

$$PA = \sqrt{\left(\frac{2}{p} - \frac{1}{p}\right)^2 + (2p)^2} = \sqrt{\left(\frac{1}{p}\right)^2 + (2p)^2}$$

$$PB = \sqrt{\left(\frac{1}{p} - 4p^3 - \frac{1}{p}\right)^2 + (2p^2)} = \sqrt{(4p^3)^2 + (2p)^2}$$

$$PA = \sqrt{\left(\frac{1}{p}\right)^2 + (2p)^2}$$

$$= \sqrt{\frac{1}{p^2} + \frac{4p^2}{1}}$$

$$= \sqrt{\frac{1+4p^4}{p^2}} = \sqrt{\frac{1}{p^2}} \times \sqrt{1+4p^4} = \frac{1}{p}\sqrt{1+4p^4} \quad (1)$$

$$PB = \sqrt{(4p^3)^2 + (2p)^2}$$

$$= \sqrt{16p^6 + 4p^2} = \sqrt{4p^2(4p^4+1)}$$

$$= \sqrt{4p^2} \times \sqrt{4p^4+1}$$

$$= 2p\sqrt{4p^4+1} \quad (2)$$

Comparing (1) and (2)

ratio $PA : PB$ is $\frac{1}{p} : 2p$ ($\times$ by p)

$1 : 2p^2$ as required