

OCR

Oxford Cambridge and RSA

Tuesday 14 June 2022 – Afternoon

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Time allowed: 2 hours



1 In this question you must show detailed reasoning.

Solve the following equations.

(a) $\frac{x}{x+1} - \frac{x-1}{x+2} = 0$

[3]

$$\frac{x}{x+1} = \frac{x-1}{x+2}$$

$$x(x+2) = (x+1)(x-1)$$

$$x^2 + 2x = x^2 - 1$$

$$2x = -1$$

$$x = -\frac{1}{2}$$

(b) $\frac{8}{x^6} - \frac{7}{x^3} - 1 = 0$

[3]

$$\frac{8 - 7x^3}{x^6} = 1$$

$$8 - 7x^3 = x^6$$

$$x^6 + 7x^3 - 8 = 0$$

$$(x^3 + 8)(x^3 - 1) = 0$$

$$x^3 = -8$$

$$x = -2$$

$$x^3 = 1$$

$$x = 1$$

$$(c) \quad 3^{x^2-7} = \frac{1}{243}$$

[2]

$$3^{x^2-7} = 3^{-5}$$

$$243 = 3^5$$
$$\frac{1}{243} = 3^{-5}$$

$$\therefore x^2 - 7 = -5$$

$$x^2 = 2$$

$$x = \pm \sqrt{2}$$

2 The points A and B have position vectors $3\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ respectively.

(a) Find the length of AB .

[2]

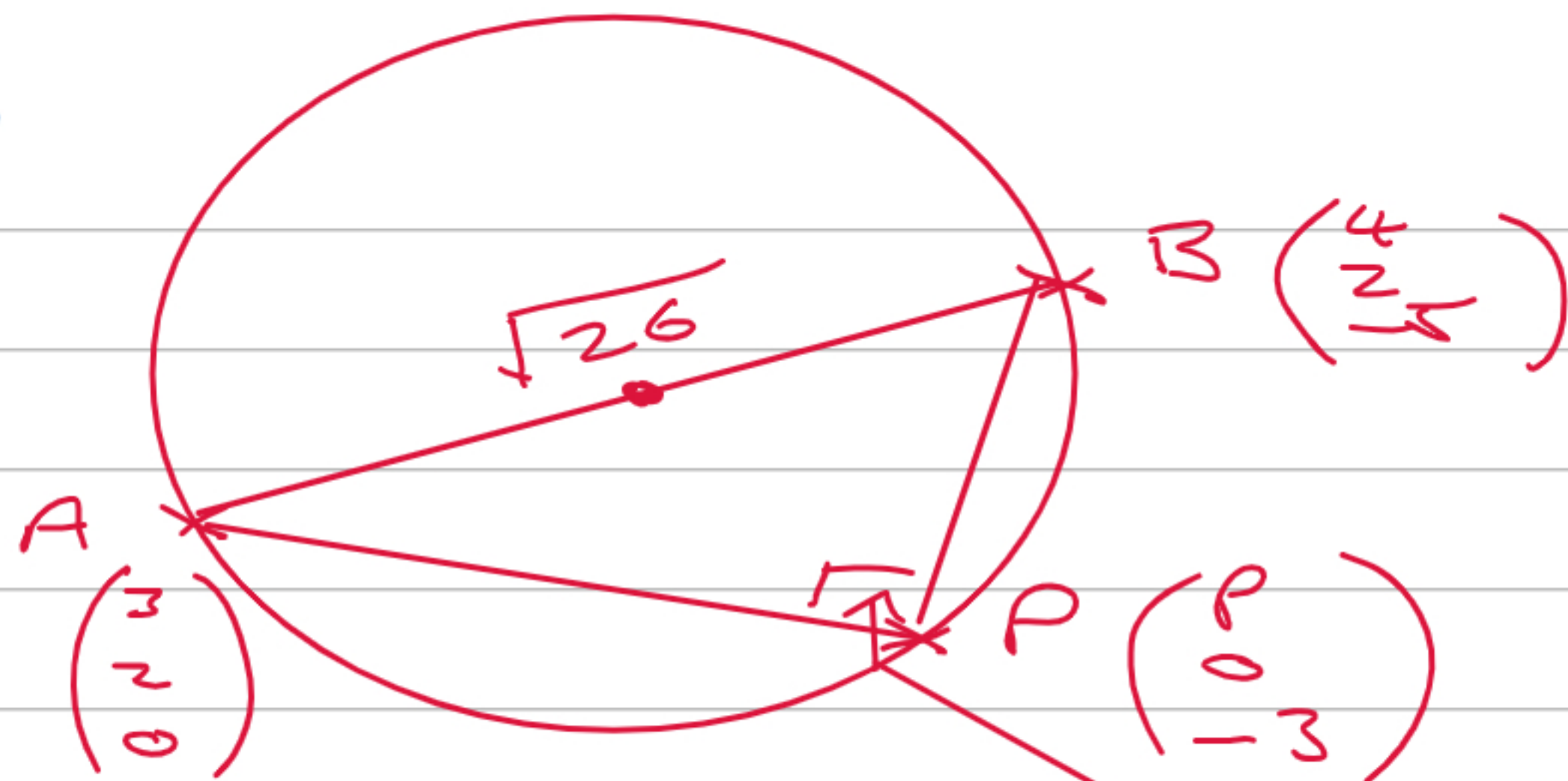
$$A = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$$

$$\begin{aligned} |\vec{AB}| &= \sqrt{(3-4)^2 + (2-2)^2 + (0-(-5))^2} \\ &= \sqrt{26} \end{aligned}$$

Point P has position vector $p\mathbf{i} - 3\mathbf{k}$, where p is a constant. P lies on the circumference of a circle of which AB is a diameter.

(b) Find the two possible values of p .

[3]



Pythagoras

$$AP^2 + BP^2 = 26$$

$$(p-0)^2 + (0-2)^2 + (-3-0)^2 + (p-4)^2 + (0-2)^2 + (-3-5)^2 = 26$$

$$p^2 - 6p + 9 + 4 + 9 + p^2 - 8p + 16 + 4 + 4 = 26$$

$$2p^2 - 14p + 46 = 26$$

$$2p^2 - 14p + 20 = 0$$

Circle theorem,
 $\angle APB$ is 90°
(angle in semicircle)

$$p^2 - 7p + 10 = 0$$

$$(p - 2)(p - 5) = 0$$

$$p = 2 \quad \text{or} \quad p = 5$$

3 (a) Amaya and Ben integrated $(1+x)^2$, with respect to x , using different methods, as follows.

Amaya: $\int (1+x)^2 dx = \frac{(1+x)^3}{3} + c = \frac{1}{3} + x + x^2 + \frac{1}{3}x^3 + c$

Ben: $\int (1+x)^2 dx = \int (1+2x+x^2) dx = x + x^2 + \frac{1}{3}x^3 + c$

Charlie said that, because these answers are different, at least one of them must be wrong.

Explain whether you agree with Charlie's statement.

[1]

Both can be correct. Amaya's has $\frac{1}{3} + c$ as a constant, whereas Ben just has c .
 The c 's are just different

(b) You are given that a is a constant greater than 1.

(i) Find $\int_1^a \frac{1}{(1+x)^2} dx$, giving your answer as a single fraction in terms of the constant a . [3]

$$\begin{aligned}
 & \int_1^a (1+x)^{-2} dx \\
 &= \left[-(1+x)^{-1} \right]_1^a = \left[-\frac{1}{1+x} \right]_1^a \\
 &= \left(-\frac{1}{1+a} \right) - \left(-\frac{1}{1+1} \right) = -\frac{1}{1+a} + \frac{1}{2} \\
 &= \frac{-2 + (1+a)}{2(1+a)} = \frac{a-1}{2(1+a)}
 \end{aligned}$$

- (ii) You are given that the area enclosed by the curve $y = \frac{1}{(1+x)^2}$, the x -axis and the lines $x = 1$ and $x = a$ is equal to $\frac{1}{3}$.

Determine the value of a .

[2]

$$\frac{a-1}{2(1+a)} = \frac{1}{3}$$

$$a-1 = \frac{2(1+a)}{3}$$

$$3(a-1) = 2+2a$$

$$3a-3 = 2+2a$$

$$a = 5$$

(c) In this question you must show detailed reasoning.

Reverse Chain Rule

Find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{\cos 2x}{\sin 2x + 2} dx$, giving your answer in its simplest form. [4]

let $I = \int \frac{\cos 2x}{\sin 2x + 2} dx$

Consider $y = \ln(\sin 2x + 2)$

$$\frac{dy}{dx} = \frac{1 \times 2 \cos 2x}{\sin 2x + 2}$$

$$\therefore I = \frac{1}{2} \ln(\sin 2x + 2) + C$$

$$\left[\frac{1}{2} \ln(\sin 2x + 2) \right]_0^{\frac{\pi}{12}}$$

$$= \left(\frac{1}{2} \ln\left(\sin\left(\frac{\pi}{6}\right) + 2\right) \right) - \left(\frac{1}{2} \ln(\sin(0) + 2) \right)$$

$$= \left(\frac{1}{2} \times \ln\left(\frac{1}{2} + 2\right) \right) - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \ln \frac{5}{2} - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \left(\ln \frac{5}{2} - \ln 2 \right)$$

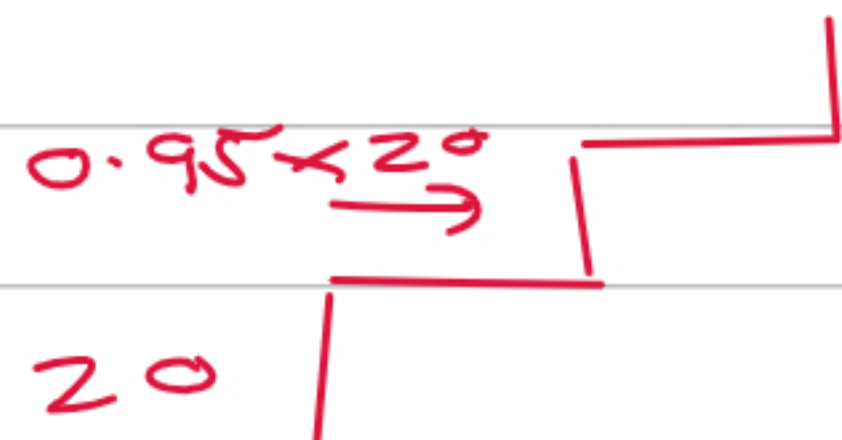
$$= \frac{1}{2} \ln \left(\frac{\frac{5}{2}}{2} \right) = \frac{1}{2} \ln \frac{5}{4}$$

- 4 An artist is creating a design for a large painting. The design includes a set of steps of varying heights. In the painting the lowest step has height 20 cm and the height of each other step is 5% less than the height of the step immediately below it.

In the painting the total height of the steps is 205 cm, correct to the nearest centimetre.

Determine the number of steps in the design.

[5]



$$a = 20 \quad r = 0.95$$

$$a \quad ar \quad ar^2 \quad ar^{n-1}$$

20 19

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$205 = \frac{20(1 - 0.95^n)}{1 - 0.95}$$

$$\frac{200 \times 0.05}{20} = 1 - 0.95^n$$

$$0.5 = 1 - 0.95^n$$

$$0.95^n = 1 - 0.5$$

$$0.95^n = 0.5$$

$$\ln 0.95^n = \ln 0.5$$

$$n \log 0.95 = \ln 0.5$$

$$n = \frac{\ln 0.5}{\ln 0.95} = 13.51$$

$$n = 14 \quad (\text{nearest whole number})$$

5 In this question you must show detailed reasoning.

A curve has equation $y = x^3 - 3x^2 + 4x$.

(a) Show that the curve has no stationary points.

[2]

$$y = x^3 - 3x^2 + 4x$$

$$\frac{dy}{dx} = 3x^2 - 6x + 4 = 0 \quad \text{at stationary pts}$$

$$b^2 - 4ac = (-6)^2 - 4 \times 3 \times 4 = -12$$

as $b^2 - 4ac < 0$ there are no real roots

so curve has no stationary points

(b) Show that the curve has exactly one point of inflection. $\frac{d^2y}{dx^2}$ changes sign [2]

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$0 = 6x - 6$$

$x = 1$ is a point of inflection

at $x = 0.9$ $\frac{dy}{dx} = 3 \times 0.9^2 - 6 \times 0.9 + 4 = 1.03$

$x = 1.1$ $\frac{dy}{dx} = 3 \times 1.1^2 - 6 \times 1.1 + 4 = 1.03$

either side of $x = 1$ gradient does not change sign, so $x = 1$ is point of inflection

6 (a) The diagrams show five different graphs. In each case the whole of the graph is shown.

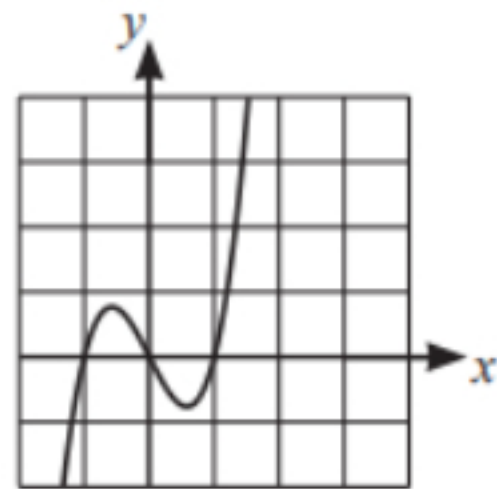


Fig. 1.1

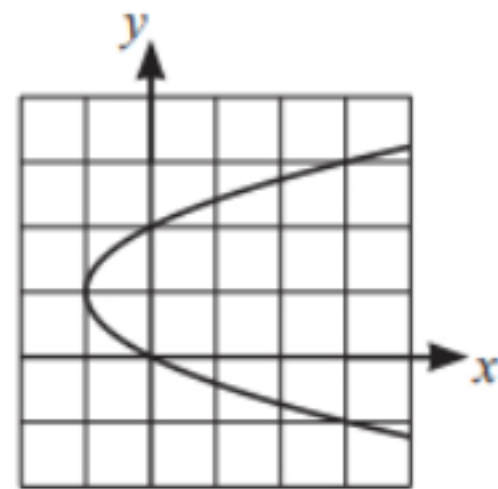


Fig. 1.2

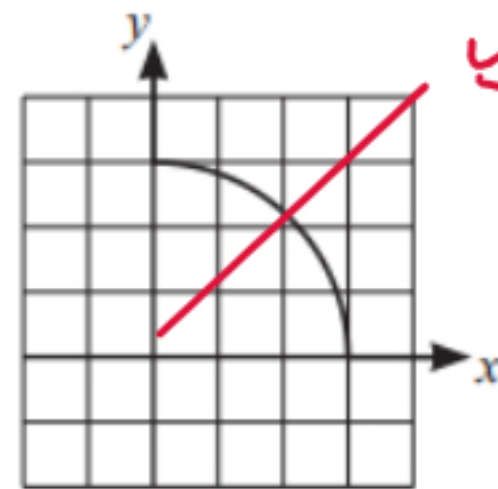


Fig. 1.3

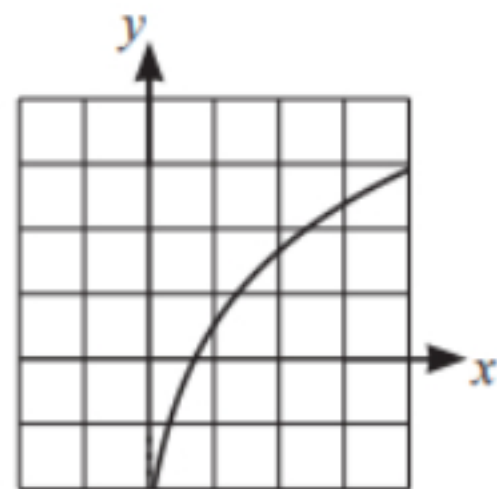


Fig. 1.4

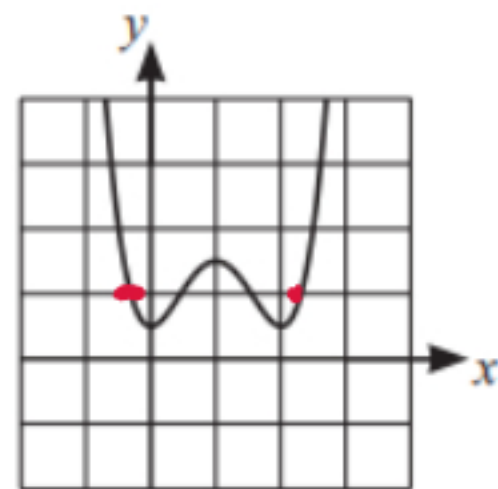


Fig. 1.5

6(a)

Graph	One-one function	Many-one function	A function that is its own inverse	Not a function
Fig. 1.1		✓		
Fig. 1.2				✓
Fig. 1.3	✓		✓	
Fig. 1.4	✓			
Fig. 1.5		✓		

one x to one y
reflect in $y=c$

x and $-x$ mapped to same y

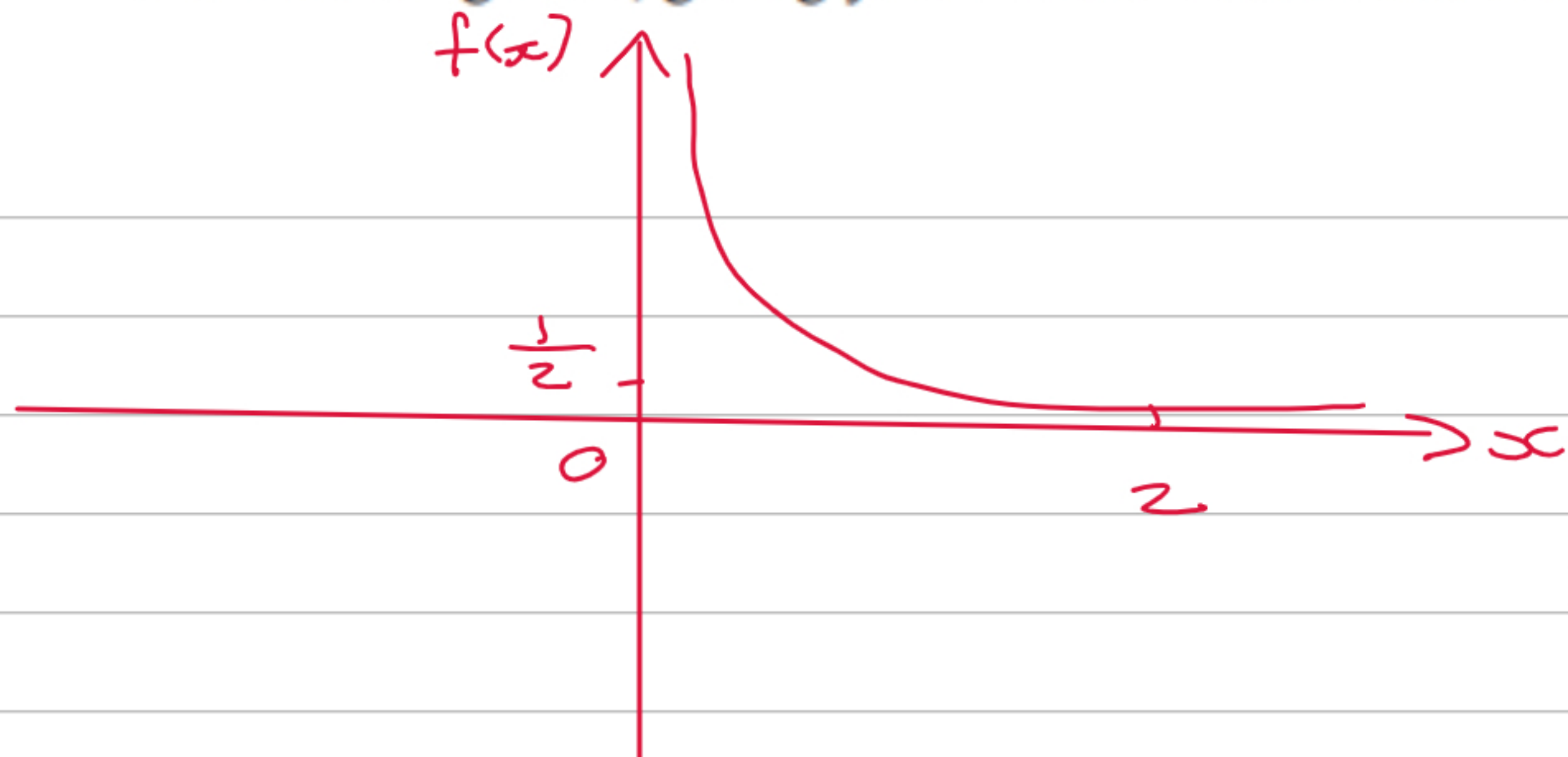
Place ticks in the boxes in the table in the Printed Answer Booklet to indicate, for each graph, whether it represents a one-one function, a many-one function, a function that is its own inverse or it does not represent a function. There may be more than one tick in some rows or columns of the table.

[4]

(b) A function f is defined by $f(x) = \frac{1}{x}$ for the domain $\{x: 0 < x \leq 2\}$.

State the range of f , giving your answer in set notation.

[2]



$$x = 2$$

$$f(x) = \frac{1}{2}$$

$$\left[\frac{1}{2}, \infty \right)$$

or

$$\left\{ y : \frac{1}{2} \leq y < \infty \right\}$$

or

$$\left\{ y : y \geq \frac{1}{2} \right\}$$

- 7 It is given that any integer can be expressed in the form $3m+r$, where m is an integer and r is 0, 1 or 2.

Use this fact to answer the following.

- (a) By considering the different values of r , prove that the square of any integer cannot be expressed in the form $3n+2$, where n is an integer. [4]

$$r = 0 \quad (3m + 0)^2 = (3m)^2 = 9m^2 = 3(3m^2) + 0$$

$$r = 1 \quad (3m + 1)^2 = 9m^2 + 6m + 1$$

$$= 3(3m^2 + 2m) + 1$$

$$r = 2 \quad (3m + 2)^2 = 9m^2 + 12m + 4$$

$$= 3(3m^2 + 4m) + 4$$

None is in form $3n + 2$

(b) Three integers are chosen at random from the integers 1 to 99 inclusive. The three integers are not necessarily different.

By considering the different values of r , determine the probability that the sum of these three integers is divisible by 3. [4]

$r = 0, 1$ or 2

if divide 1 to 99 by 3, remainder will be 0, 1 or 2 and sum of 3 remainders is 0, 3, 6 if the sum is divisible by 3

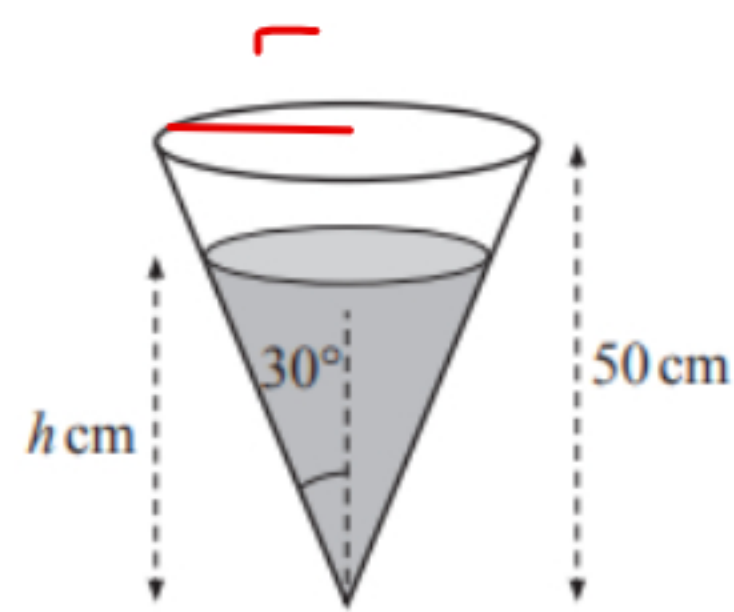
Combinations of 3 remainders are

0 0 0 ✓	1 0 0	2 0 0
0 0 1	1 0 1	2 0 1 ✓
0 0 2	1 0 2 ✓	2 0 2
0 1 0	1 1 0	2 1 0 ✓
0 1 1	1 1 1 ✓	2 1 1
0 1 2 ✓	1 1 2	2 1 2
0 2 0	1 2 0 ✓	2 2 0
0 2 1 ✓	1 2 1	2 2 1
0 2 2	1 2 2	2 2 2 ✓

probability

$$= \frac{9}{27}$$

(9 ticked combinations)



Variables
 V, h, t

The diagram shows a water tank which is shaped as an inverted cone with semi-vertical angle 30° and height 50 cm. Initially the tank is full, and the depth of the water is 50 cm.

Water flows out of a small hole at the bottom of the tank. The rate at which the water flows out is modelled by $\frac{dV}{dt} = -2h$, where $V \text{ cm}^3$ is the volume of water remaining and $h \text{ cm}$ is the depth of water in the tank t seconds after the water begins to flow out.

Determine the time taken for the tank to become empty.

[For a cone with base radius r and height h the volume V is given by $\frac{1}{3}\pi r^2 h$.]

[7]

$$\therefore V = \frac{1}{3} \times \pi \times \left(\frac{h\sqrt{3}}{3} \right)^2 \times h$$

$$V = \frac{1}{9} \pi h^3$$

differentiating

$$\frac{dV}{dh} = \frac{3}{9} \times \pi h^2 = \frac{1}{3} \pi h^2$$



$$\tan 30^\circ = \frac{r}{h}$$

$$r = h \tan 30^\circ$$

$$r = h \times \frac{\sqrt{3}}{3}$$

$$\frac{dV}{dt} = -2h \quad (1)$$

linking variables V, h, t

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{1}{3} \pi h^2 \times \frac{dh}{dt}$$

① gives $\frac{dV}{dt} = -2h$

$$-2h = \frac{1}{3} \pi h^2 \times \frac{dh}{dt}$$

Separate variables then integrate

$$\int -6 dt = \int \pi h dh$$

$$-6t = \frac{1}{2} \pi h^2 + C$$

at $t=0, h=50$

$$0 = \frac{1}{2} \times \pi \times 50^2 + C$$

$$C = -1250\pi$$

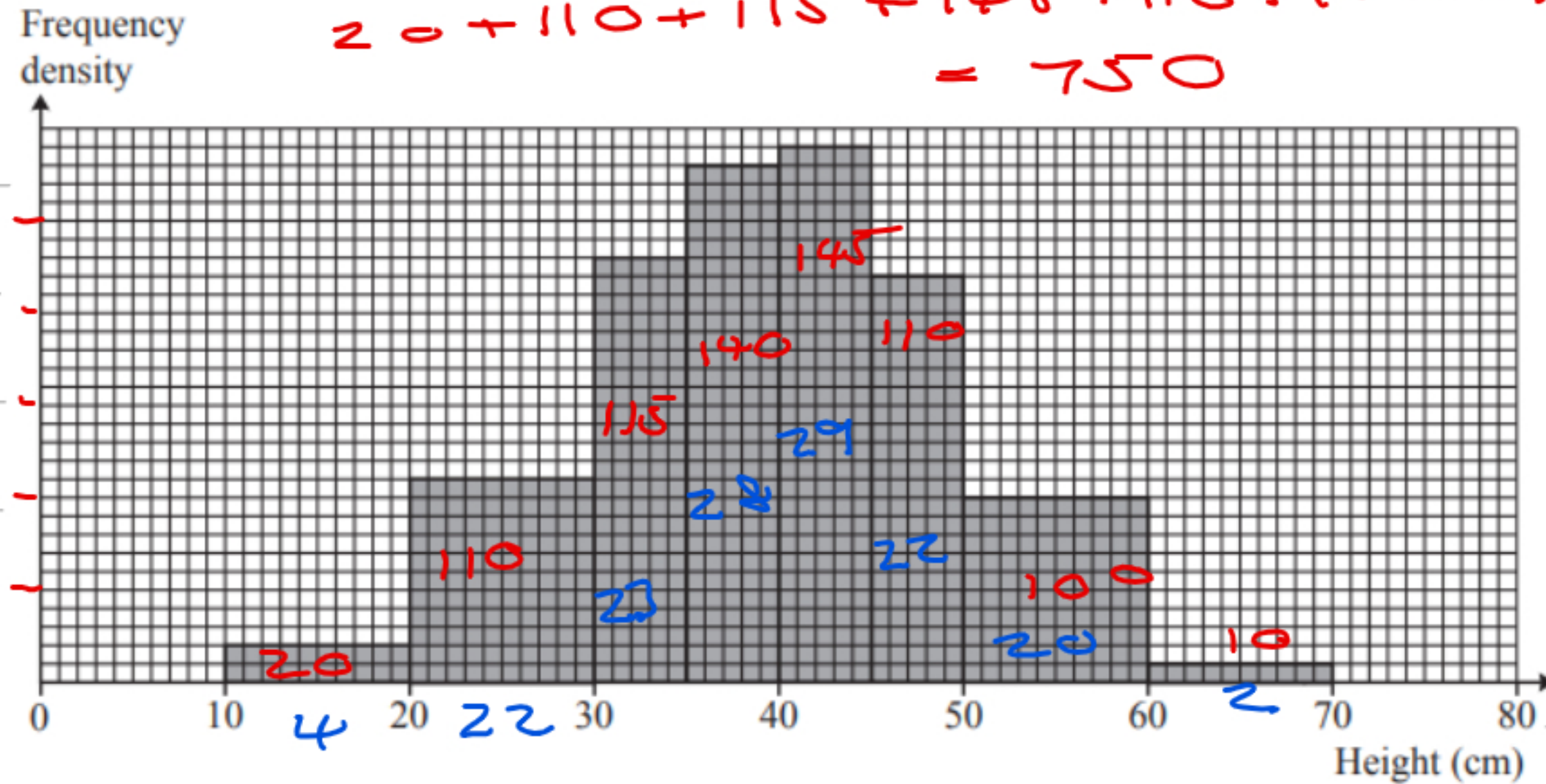
$$-6t = \frac{1}{2} \pi h^2 - 1250\pi$$

tank empty when $h = 0$

$$t = \frac{-1250\pi}{-6} = \frac{625\pi}{3} \text{ seconds}$$

Section B: Statistics
Answer all the questions.

- 9 The heights, in centimetres, of a random sample of 150 plants of a certain variety were measured. The results are summarised in the histogram.



One of the 150 plants is chosen at random, and its height, X cm, is noted.

- (a) Show that $P(20 < X < 30) = 0.147$, correct to 3 significant figures.

[2]

Sam suggests that the distribution of X can be well modelled by the distribution $N(40, 100)$.

$$P(20 < X < 30) = \frac{110}{750} \approx 0.146$$

$$= 0.147 \text{ (3 s.f.)}$$

750 squares represents

150 plants

\therefore frequency $\div 5$

Sam suggests that the distribution of X can be well modelled by the distribution $N(40, 100)$.

(b) (i) Give a brief justification for the use of the normal distribution in this context. [1]

(ii) Give a brief justification for the choice of the parameter values 40 and 100. [2]

(i) It is bell shaped, peaks in middle, tails off at each side and roughly symmetrical

(ii) Roughly symmetrical about $x = 40$ (mean) μ

$$100 = \sigma^2 \quad \therefore \sigma = 10$$

If we go σ to right of mean

$$145 + 110 = 255$$

σ to left of mean

$$140 + 115 = 255$$

$$\frac{255 + 255}{750} = \frac{510}{750} = 0.68$$

∴ 68% within one standard deviation of mean which is fitting for normal distribution

(c) Use Sam's model to find $P(20 < X < 30)$.

[1]

Mean 7

$$p = 0.1359$$

Normal (d)

$$\text{Low} = 20$$

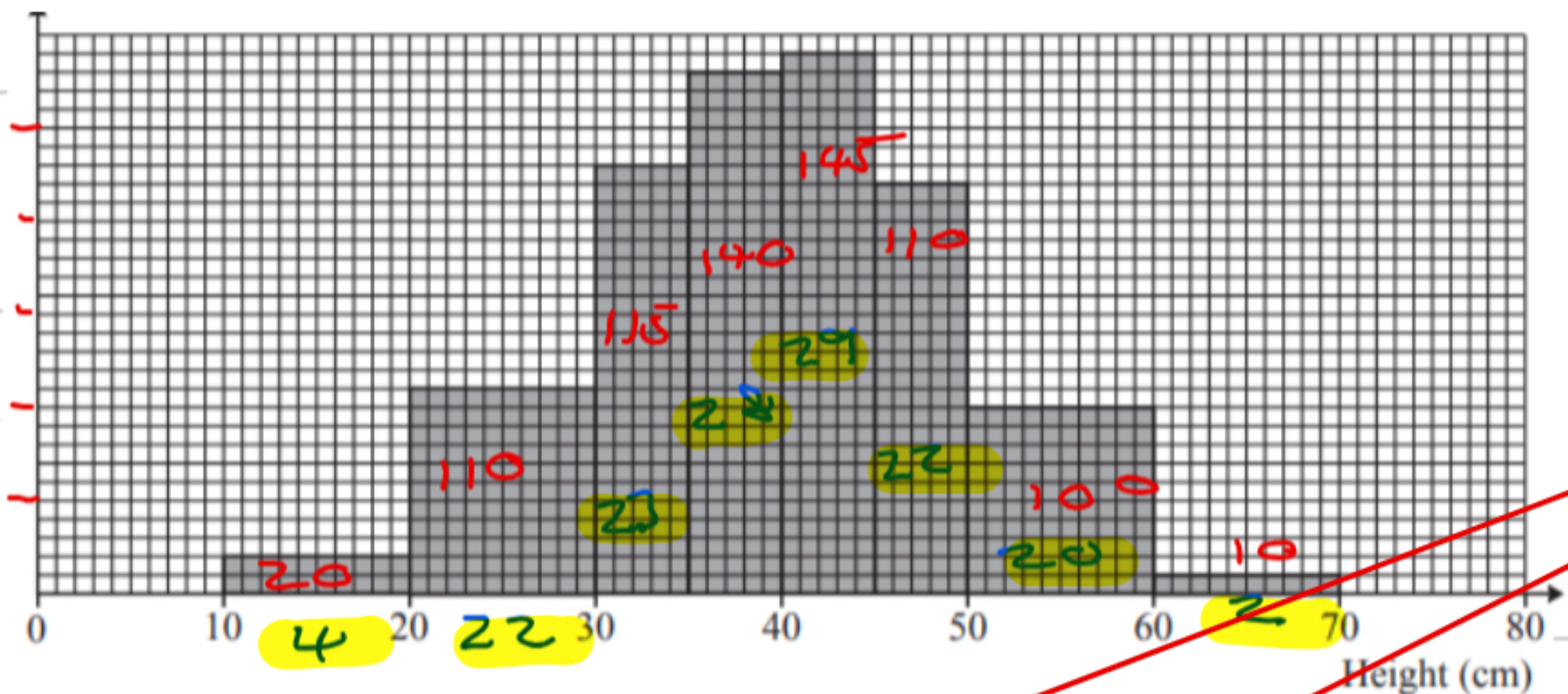
$$\text{Upper} = 30$$

$$\sigma = 10$$

$$\mu = 40$$

Nina suggests a different model. She uses the midpoints of the classes to calculate estimates, m and s , for the mean and standard deviation respectively, in centimetres, of the 150 heights. She then uses the distribution $N(m, s^2)$ as her model.

(d) Use Nina's model to find $P(20 < X < 30)$.



$\mu = \bar{x} = 39.4$
 $\sigma = 10.3$

mid pt
frequency

[4]

9(e)(i)

x	Below 20	20 to 30	30 to 35	35 to 40	40 to 45	45 to 50	50 to 60	Above 60
Probability obtained from histogram	0.027	0.147	0.153	0.187	0.193	0.147	0.133	0.013
Probability obtained from Sam's model, $N(40, 100)$	0.023	0.136	0.150	0.191	0.197	0.150	0.136	0.023
Probability obtained from Nina's model, $N(m, s^2)$	0.030	0.151	0.153	0.188	0.183	0.142	0.130	0.023

\bar{x}
 $\sigma^2 = 100$

Normal CD
 $\mu = \text{height}$
 $\sigma = 10$

Menu 6

Option 3: 1 var calc

1: 1 variable

x	freq	x	freq
42.5	29	15	4
47.5	22	25	22
55	20	32.5	23
65	2	37.5	28

(ii) By considering the different ranges of values of X given in the table, discuss how well the two models fit the original distribution. [2]

x	< 20	20 to 30	30 to 35	35 to 40	40 to 45	45 to 50	50 to 60	> 60
Histogram	0.027	0.147	0.153	0.187	0.193	0.147	0.133	0.013
N(40, 100)	0.023	0.136	0.150	0.191	0.191	0.150	0.136	0.023
N(μ, σ^2)	0.030	0.151	0.153	0.189	0.183	0.142	0.130	0.023

Sam

Nina

Compare Sam's and Nina's to histogram for each interval

Nina's better for $x < 40$

but less good for 40 to 45 or > 60

Sam's better for $x > 40$

but less good for 20 to 30 or > 60

- 10 The table shows the age structure of usual residents of 18 Local Authorities (LAs) in the North West region of the UK in 2011.

Local Authority	Age 0 to 17	Age 18 to 24	Age 25 to 64	Age 65 and over
A	26.20%	9.06%	51.81%	12.92%
B	23.32%	8.99%	52.32%	15.37%
C	22.24%	8.96%	52.56%	16.23%
D	22.67%	8.10%	53.27%	15.96%
E	20.70%	7.77%	54.77%	16.76%
F	18.14%	6.51%	51.13%	24.21%
G	18.96%	14.20%	48.51%	18.33%
H	19.06%	14.79%	52.12%	14.04%
I	25.15%	9.04%	51.16%	14.65%
J	22.93%	8.81%	52.22%	16.04%
K	21.48%	13.98%	50.82%	13.73%
L	23.98%	9.20%	52.26%	14.56%
M	21.67%	11.19%	52.94%	14.19%
N	17.82%	6.01%	51.93%	24.23%
O	22.83%	7.30%	53.86%	16.01%
P	21.76%	8.28%	54.03%	15.93%
Q	21.42%	8.43%	53.90%	16.25%
R	18.61%	7.33%	49.35%	24.71%

Percentage of residents

- (a) Without reference to any other columns, explain how you would use **only** the columns for the age ranges 0 to 17 and 18 to 24 to decide whether an LA might be one of the following.
- (i) An LA that includes a university
 - (ii) An LA that attracts young couples to live
 - (iii) An LA that attracts retired people to live

- [1] Higher proportion for 18-24 year olds
- [1] Higher proportion for 0-17 or 18-24 or both
- [1] Lower proportion for 0-17 or 18-24 or both

(b) Using your answers to part (a), identify the following.

(i) Four LAs that might include a university

[1] P, H, K, M

(ii) Three LAs that might be attractive to retired people

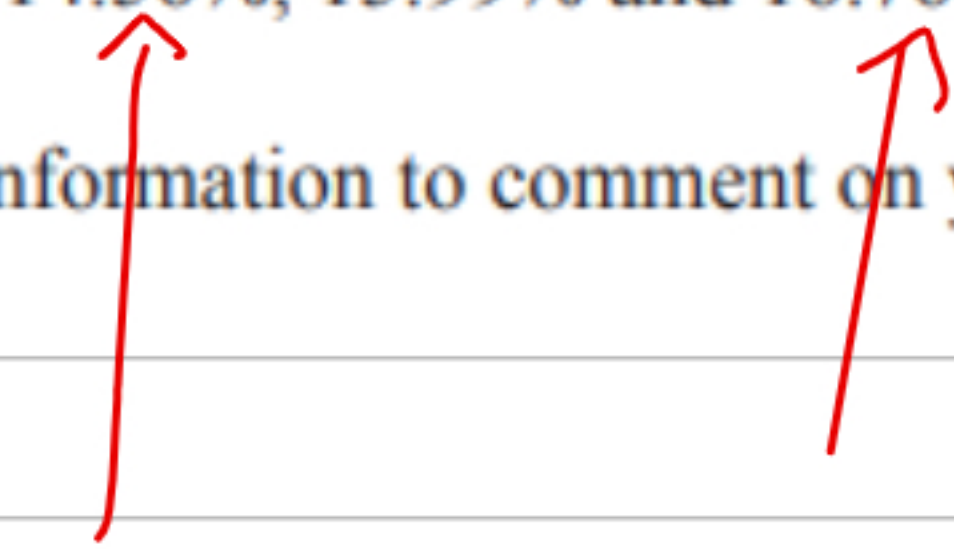
[1] F, N, R

(c) Explain why your answer to part (b)(ii), based only on the columns for the age ranges 0 to 17 and 18 to 24, may not be reliable. [1]

May be large % of 25-64 year olds
 rather than > 65,
 or low 0-17 or low 18-24

(d) The lower quartile, median and upper quartile of the percentages in the column "Age 65 and over" are 14.56%, 15.99% and 16.76% respectively.

Use this information to comment on your answers to part (b)(ii) and part (c). [2]


 LQ UQ

This means $IQR = 16.76 - 14.56 = 2.2$

$$16.76 + \underbrace{3 \times 2.2}_{+ 3 \text{ sd's}} = 20.06$$

F, N, R all > 20.06

so these are all outliers for > 65

So F, N, R OK despite (c)

In a magazine article, a councillor plans to describe a typical LA in the North West region. He wants to quote the average percentage of residents aged 65 or over.

(e) The mean of the percentages in the column "Age 65 and over" is 16.90%.

Use this information, and the information given in part (d), to explain whether the median or the mean better represents the data in the column "Age 65 and over". [2]

from d) $uQ = 16.76$
as mean $16.90\% > uQ$

better to use median

11 In the past the masses of new-born babies in a certain country were normally distributed with mean 3300 g. Last year a publicity campaign was held to encourage pregnant women to improve their diet.

Following this campaign, it is required to test whether the mean mass of new-born babies has increased. A random sample of 200 new-born babies is chosen, and it is found that their mean mass is 3360 g. It is given that the standard deviation of the masses of new-born babies is 450 g.

Carry out the test at the 2.5% significance level.

[7]

$$X \sim N(3300, 31.8198^2)$$

$$H_0: \mu = 3300 \text{ g}$$

$$H_1: \mu > 3300 \text{ g}$$

$$n = 200$$

$$\bar{X} = 3360 \text{ g}$$

amend

$$\sigma = \frac{\sigma}{\sqrt{n}} = \frac{450}{\sqrt{200}} = 31.8198$$

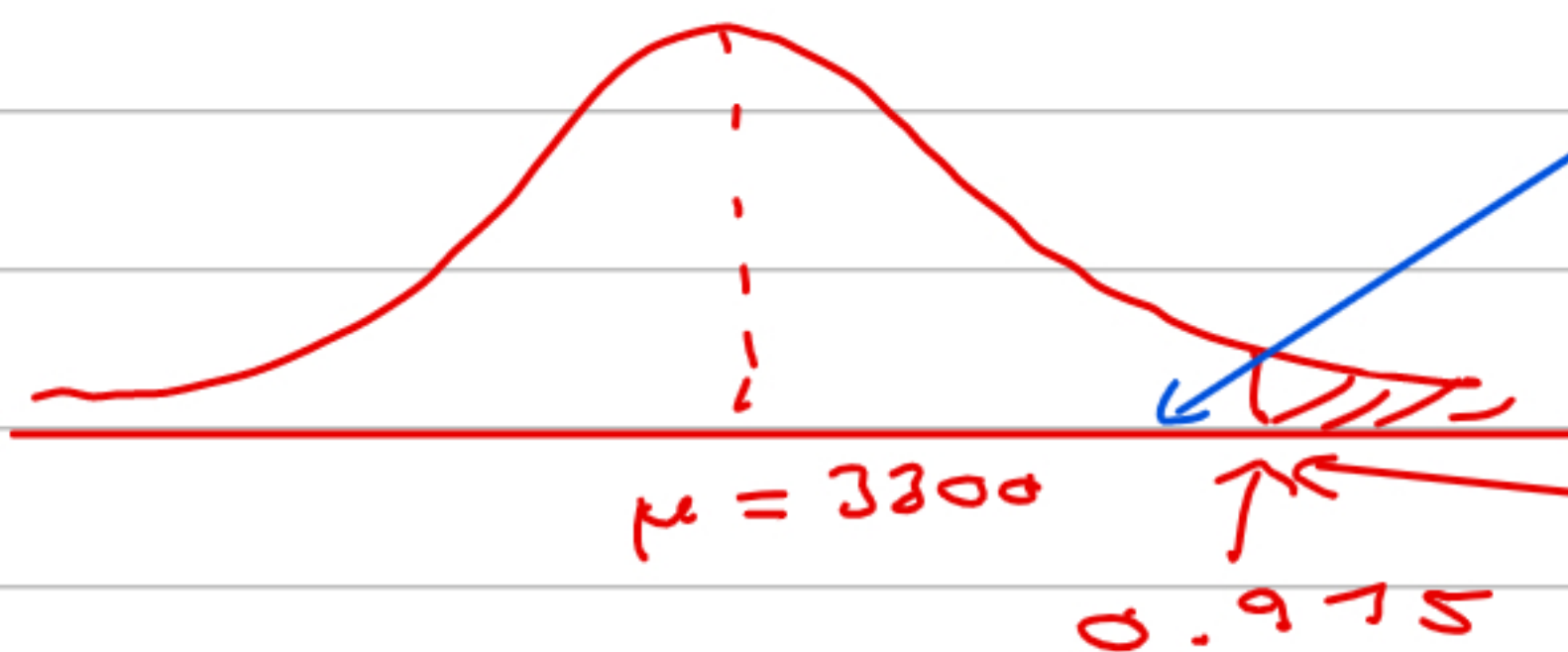
Inverse normal

$$\text{Area} = 0.975 \quad (2.5\% \text{ upper tail})$$

$$\sigma = 31.8198$$

$$\mu = 3300$$

$$\text{gives } 3362.365$$



As $\bar{X} = 3360$, $\bar{X} < 3362.365$
as not in critical region, accept H_0 , so
the campaign to improve diet has not worked

12 A firm claims that no more than 2% of their packets of sugar are underweight. A market researcher believes that the actual proportion is greater than 2%. In order to test the firm's claim, the researcher weighs a random sample of 600 packets and carries out a hypothesis test, at the 5% significance level, using the null hypothesis $p = 0.02$.

(a) Given that the researcher's null hypothesis is correct, determine the probability that the researcher will conclude that the firm's claim is incorrect. [5]

(b) The researcher finds that 18 out of the 600 packets are underweight. A colleague says "18 out of 600 is 3%, so there is evidence that the actual proportion of underweight bags is greater than 2%."

Criticise this statement. ^{b)} 18 is not in critical region, as $X \geq 19$ [1]

$$X \sim B(600, 0.02)$$

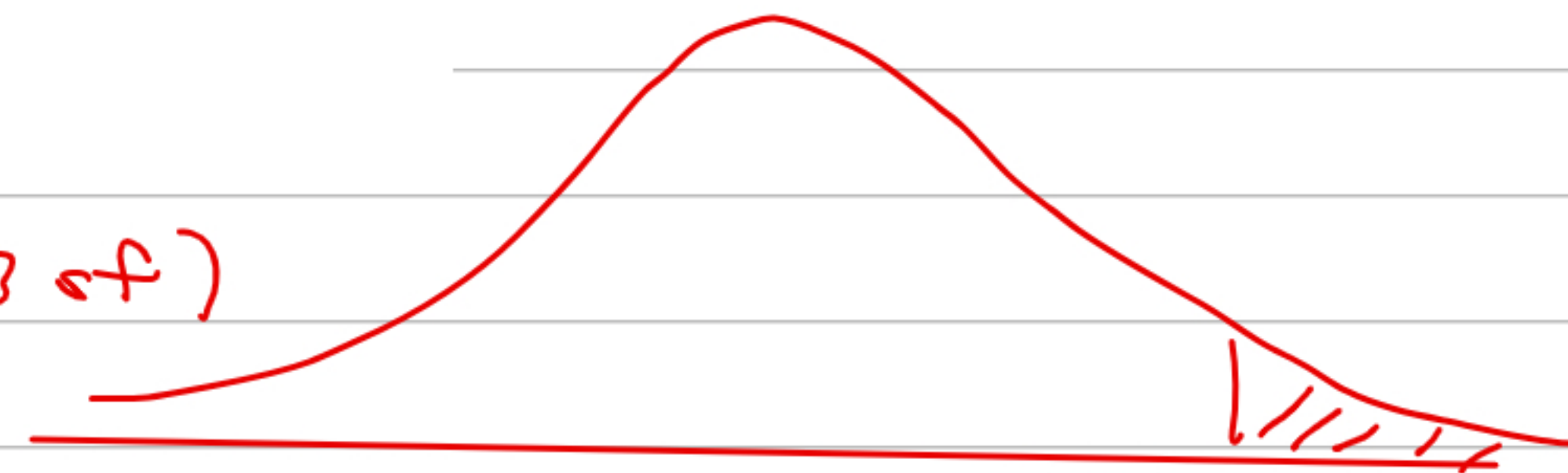
\uparrow \uparrow
 n p

$$H_0: p = 0.02$$

$$H_1: p > 0.02$$

$$\begin{aligned}
 a) \quad 1 - 0.9641 &= 0.03588 \\
 &= 0.0359 \text{ (3 sf)}
 \end{aligned}$$

Probability that claim incorrect = 0.0359 (3 sf)

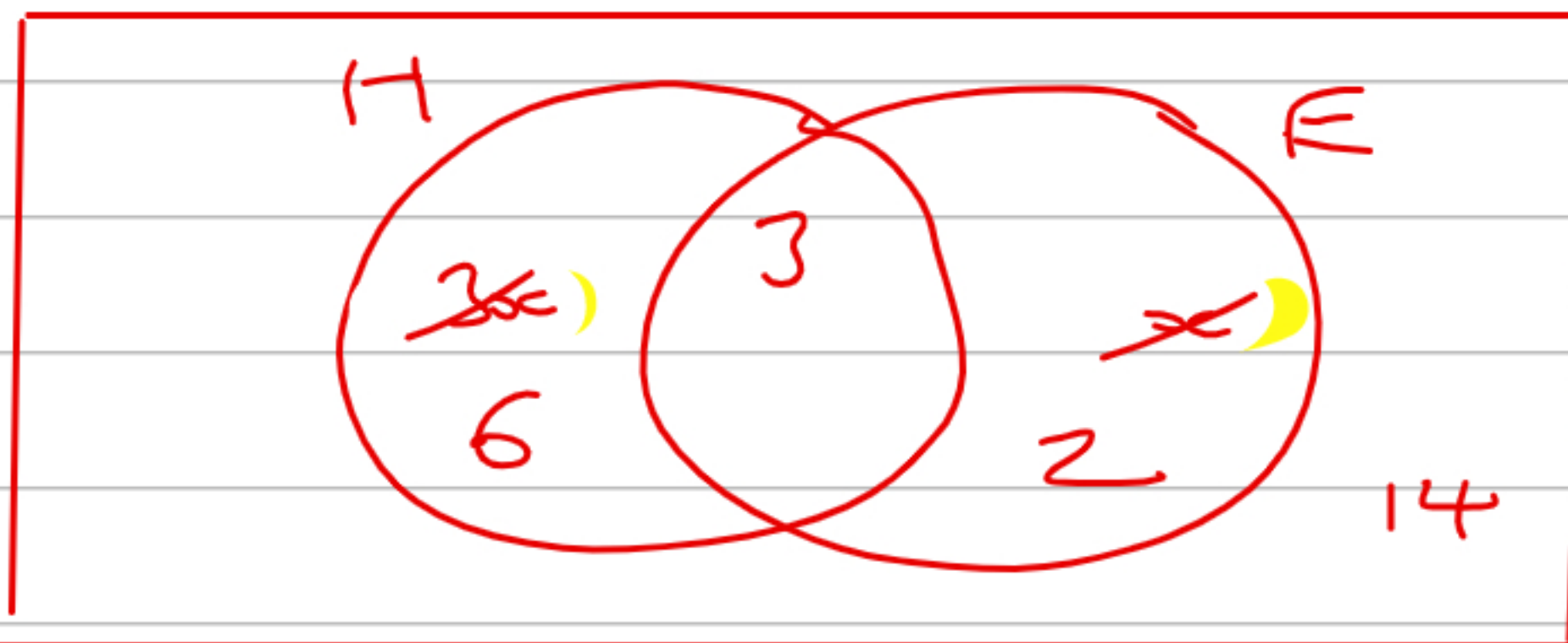


$$\begin{aligned}
 P(X \geq 17) &= 0.9389 \uparrow \\
 P(X \geq 18) &= 0.9641 \quad 0.95
 \end{aligned}$$

Critical region $X \geq 19$

13 There are 25 students in a class.

- The number of students who study both History and English is 3.
 - The number of students who study neither History nor English is 14.
 - The number of students who study History but not English is three times the number who study English but not History.
- (a)
- Show this information on a Venn diagram.
 - Determine the probability that a student selected at random studies English. [4]



$$\therefore 4x + 3 + 14 = 25$$

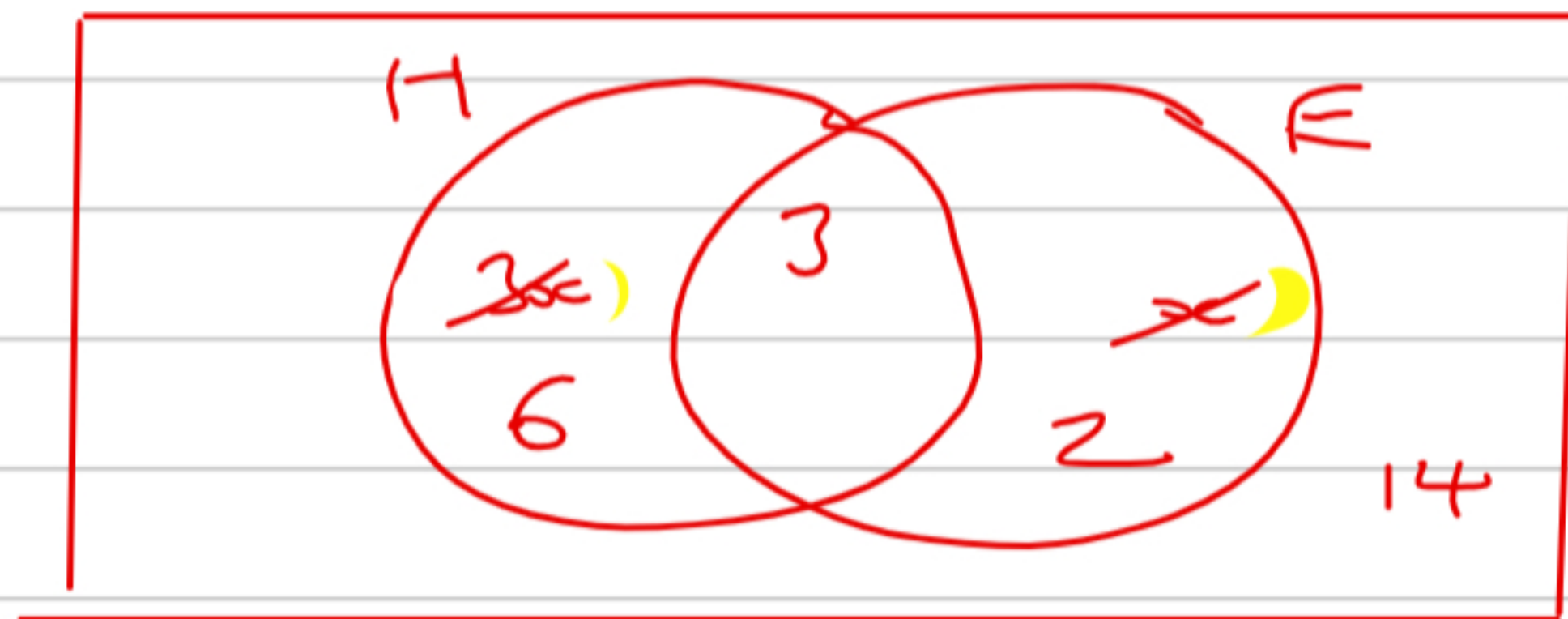
$$4x + 17 = 25$$

$$4x = 8$$

$$P(\text{English}) = \frac{5}{25}$$

Two different students from the class are chosen at random.

- (b) Given that exactly one of the two students studies English, determine the probability that exactly one of the two students studies History. [6]



$$P(H | E) = \frac{3}{3+2} = \frac{3}{5} \quad \times$$

$$P(H' | E') = \frac{14}{6+14} = \frac{14}{20}$$

↑ opposite

$$P(H' | E) = \frac{2}{3+2} = \frac{2}{5} \quad \times$$

$$P(H | E') = \frac{6}{6+14} = \frac{6}{20}$$

↑ opposite

$$\frac{3}{5} \times \frac{14}{20} + \frac{2}{5} \times \frac{6}{20}$$

$$= \frac{27}{50}$$