OCR
Oxford Cambridge and RSA

Wednesday 12 June 2019 – Morning

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Time allowed: 2 hours

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

Ctrl

1 **(a)** Differentiate the following.

(i) $\dfrac{x^2}{2x+1}$ [3]

(ii) $\tan(x^2 - 3x)$ [2]

(b) Use the substitution $u = \sqrt{x} - 1$ to integrate $\dfrac{1}{\sqrt{x}-1}$. [4]

(c) Integrate $\dfrac{x-2}{2x^2 - 8x - 1}$. [2]

1 a (i) $u = x^2$ $v = 2x + 1$ $\dfrac{vu' - uv'}{v^2}$

$u' = 2x$ $v' = 2$

quotient rule $\dfrac{2x(2x+1) - 2x^2}{(2x+1)^2} = \dfrac{4x^2 + 2x - 2x^2}{(2x+1)^2}$

$= \dfrac{2x^2 + 2x}{(2x+1)^2} = \dfrac{2x(x+1)}{(2x+1)^2}$

(ii) $\tan(x^2 - 3x)$ [2]

(ii) $\frac{d}{dx} \tan(x^2 - 3x)$

$= \sec^2(x^2 - 3x) \times (2x - 3)$

$= (2x - 3)\sec^2(x^2 - 3x)$

(b) Use the substitution $u = \sqrt{x} - 1$ to integrate $\dfrac{1}{\sqrt{x}-1}$. **[4]**

$$u = x^{\frac{1}{2}} - 1$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{\frac{1}{2}x^{-\frac{1}{2}}} = dx$$

$$2x^{\frac{1}{2}}\,du = dx$$

$$2(u+1)\,du = dx$$

$$\int \frac{1}{\sqrt{x}-1}\,dx$$

$$\int \frac{1}{u} \times 2(u+1)\,du$$

$$= \int \frac{2u+2}{u}\,du$$

$$= \int 2 + \frac{2}{u}\,du$$

$$= 2u + 2\ln u + c$$

$$= 2(\sqrt{x}-1) + 2\ln(\sqrt{x}-1) + c$$

(c) Integrate $\frac{x-2}{2x^2-8x-1}$. [2]

let $I = \int \frac{x-2}{2x^2-8x-1}\,dx$

REVERSE CHAIN RULE

let $y = \ln|2x^2-8x-1|$

$$\frac{dy}{dx} = \frac{4x-8}{2x^2-8x-1} = \frac{4(x-2)}{2x^2-8x-1}$$

$$\therefore \int \frac{x-2}{2x^2-8x-1}\,dx = \frac{1}{4}\ln|2x^2-8x-1| + c$$

2 **(a)** Find the coefficient of x^5 in the expansion of $(3-2x)^8$. **[1]**

(b) **(i)** Expand $(1+3x)^{0.5}$ as far as the term in x^3. **[3]**

(ii) State the range of values of x for which your expansion is valid. **[1]**

a)

$$(3-2x)^8 = 3^8\left(1-\frac{2}{3}x\right)^8$$

$$x^5 \text{ term} \quad 3^8 \times \frac{8\times7\times6\times5\times4}{1\times2\times3\times4\times5} \times \left(-\frac{2}{3}\right)^5$$

$$= 6561 \times \frac{6720}{120} \times -\frac{32}{243}$$

$$= -48384$$

(b) **(i)** Expand $(1+3x)^{0.5}$ as far as the term in x^3. **[3]**

(ii) State the range of values of x for which your expansion is valid. **[1]**

$$(1+3x)^{\frac{1}{2}}$$

$$= 1 + \frac{\frac{1}{2}}{1}(3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2}(3x)^2$$

$$+ \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{1 \times 2 \times 3}(3x)^3$$

$$= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \ldots$$

valid for $|x| < \frac{1}{3}$

$$-\frac{1}{3} < x < \frac{1}{3}$$

A student suggests the following check to determine whether the expansion obtained in part **(b)(i)** may be correct.

"Use the expansion to find an estimate for $\sqrt{103}$, correct to five decimal places, and compare this with the value of $\sqrt{103}$ given by your calculator."

(iii) Showing your working, carry out this check on your expansion from part **(b)(i)**. **[3]**

$$(1+3x)^{\frac{1}{2}}$$

$$\frac{\sqrt{103}}{\sqrt{100}} = \sqrt{\frac{103}{100}} = \sqrt{1.03}$$

if $x = 0.01$ expansion valid $-\frac{1}{3} < x < \frac{1}{3}$

$$1 + \frac{3}{2} \times 0.01 - \frac{9}{8} \times (0.01)^2 + \frac{27}{16}(0.01)^3 = 1.01488918 8$$

Using expansion $\therefore$ $\sqrt{103} = 10 \times 1.014889188 = 10.14889$ (5dp)

Using calculator $\sqrt{103} = 10.14889$ to 5dp as well

so expansion is correct

3 (a) A circle is defined by the parametric equations $x = 3 + 2\cos\theta$, $y = -4 + 2\sin\theta$.

(i) Find a cartesian equation of the circle. [2]

(ii) Write down the centre and radius of the circle. [1]

a i) $\frac{x-3}{2} = \cos\theta$ $\frac{y+4}{2} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{y+4}{2}\right)^2 + \left(\frac{x-3}{2}\right)^2 = 1$

$(y+4)^2 + (x-3)^2 = 4$

ii) centre $(3, -4)$ radius $= 2$

(b) In this question you must show detailed reasoning.

The curve S is defined by the parametric equations $x = 4\cos t$, $y = 2\sin t$. The line L is a tangent to S at the point given by $t = \frac{1}{6}\pi$.

Find where the line L cuts the x-axis. — at $y = 0$ **[6]**

$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 2\cos t$$

$$\frac{dy}{dx} = \frac{2\cos t}{-4\sin t} = -\frac{1}{2}\cot t = -\frac{1}{2\tan t}$$

at $t = \frac{1}{6}\pi$, $\frac{dy}{dx} = -\frac{\sqrt{3}}{2}$

at $t = \frac{\pi}{6}$, $x = 4\cos\frac{\pi}{6} = 2\sqrt{3}$

$y = 2\sin\frac{\pi}{6} = 1$

$y - y_1 = m(x - x_1)$

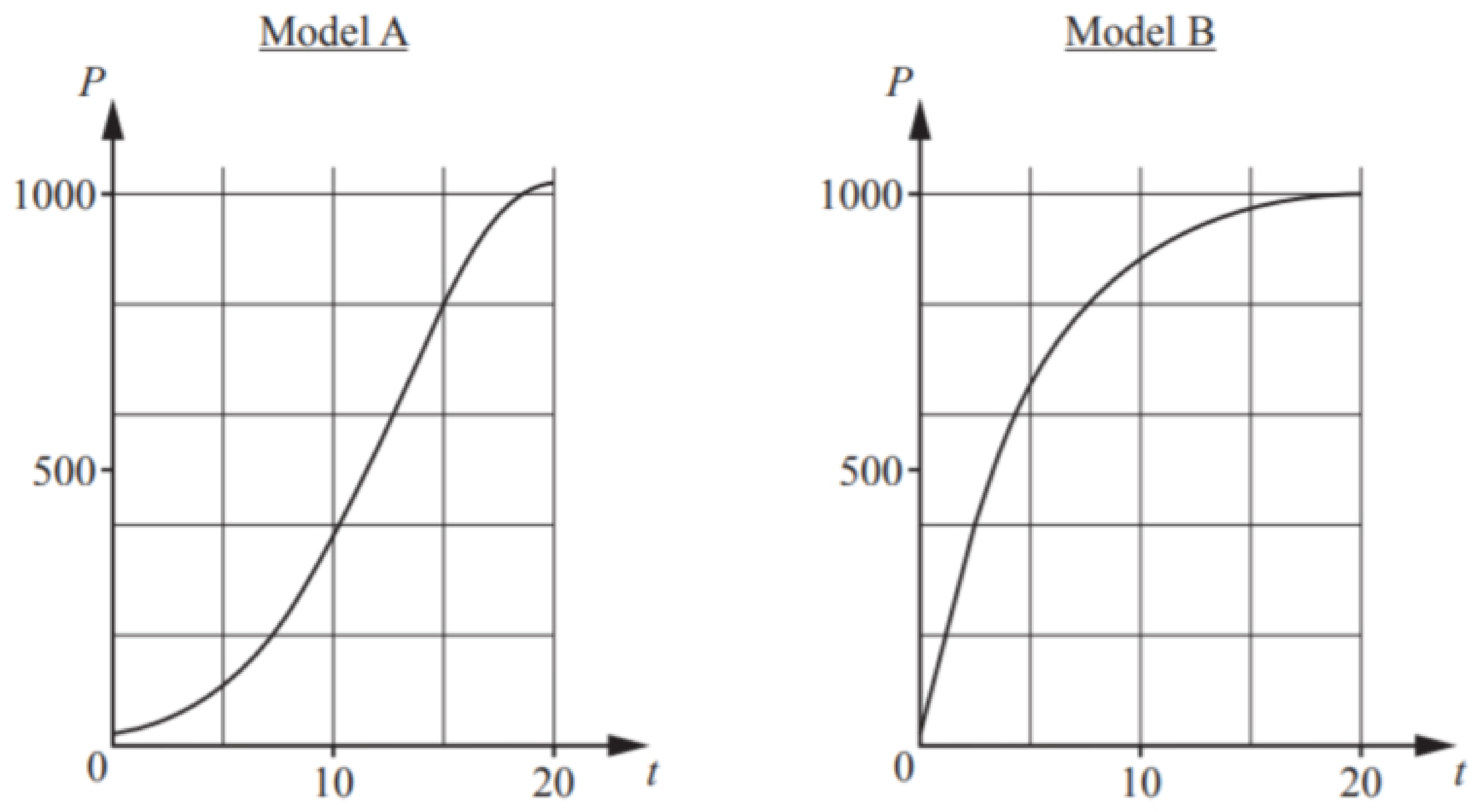
L is $y - 1 = -\frac{\sqrt{3}}{2}(x - 2\sqrt{3})$

at $y = 0$ $\quad -1 = -\frac{\sqrt{3}}{2}x + 3$

$\frac{\sqrt{3}}{2}x = 4 \qquad x = \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$

Cuts x-axis at $\left(\frac{8\sqrt{3}}{3}, 0\right)$

4 A species of animal is to be introduced onto a remote island. Their food will consist only of various plants that grow on the island. A zoologist proposes two possible models for estimating the population P after t years. The diagrams show these models as they apply to the first 20 years.



(a) Without calculation, describe briefly how the rate of growth of P will vary for the first 20 years, according to each of these two models. **[1]**

a) Growth rate grows slowly, then quickly, then slowly

The equation of the curve for model A is $P = 20 + 1000e^{-\frac{(t-20)^2}{100}}$.

The equation of the curve for model B is $P = 20 + 1000\left(1 - e^{-\frac{t}{5}}\right)$.

(b) Describe the behaviour of P that is predicted for $t > 20$

(i) using model A, **[1]**

(ii) using model B. **[1]**

(i) $P = 20 + 1000e^{-0}$ at $t = 20$

$P = 20 + 1000e^{-1}$ at $t = 30$

$\therefore$ P is 1020 at $t = 20$, then as t increases P tends to 20

(ii) $P = 20 + 1000 - 1000e^{-4}$ at $t = 20$

then as t increases, P decreases, doesn't exceed 1020

There is only a limited amount of food available on the island, and the zoologist assumes that the size of the population depends on the amount of food available and on no other external factors.

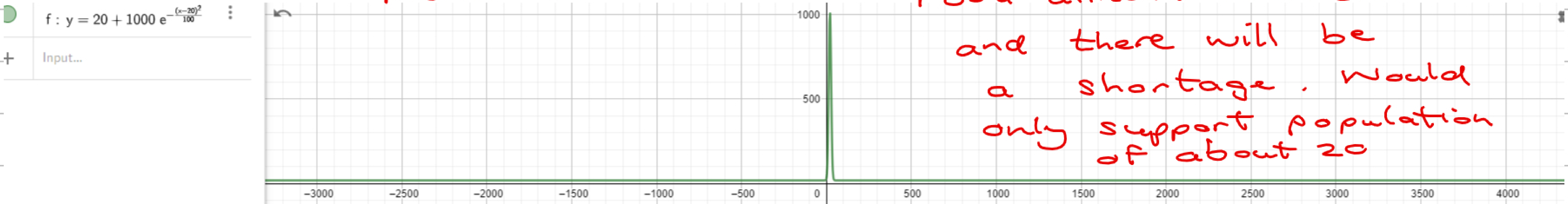
(c) State what is suggested about the long-term food supply by

(i) model A, **[1]**

(ii) model B. **[1]**

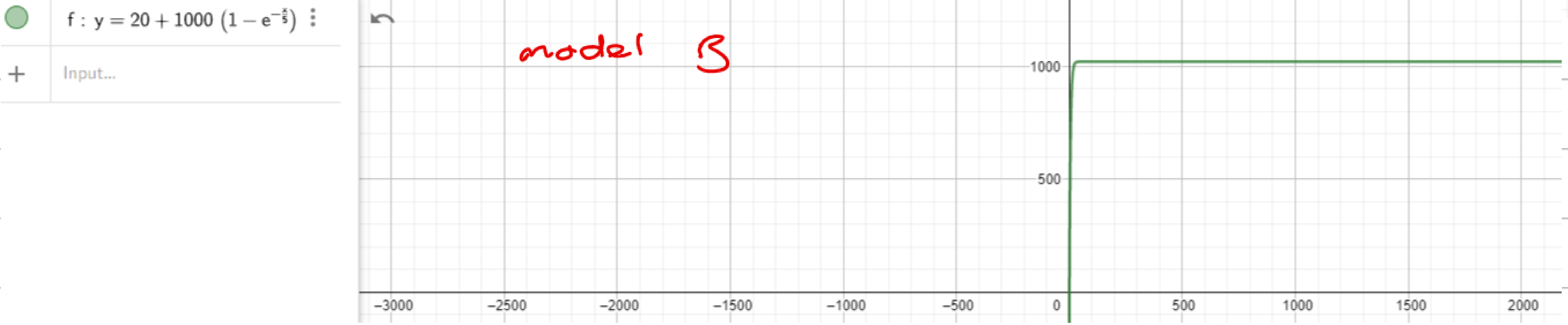
(i)

model A



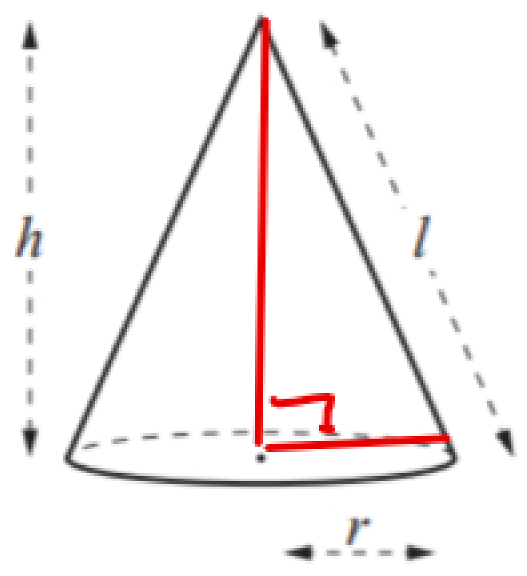
Food almost runs out and there will be a shortage. Would only support population of about 20

model B



Enough food to support a population of ≈ 1020

5



For a cone with base radius r, height h and slant height l, the following formulae are given.

Curved surface area, $S = \pi r l$

Volume, $V = \frac{1}{3}\pi r^2 h$

A container is to be designed in the shape of an inverted cone with no lid. The base radius is r m and the volume is V m^3.

The area of the material to be used for the cone is 4π m^2.

(a) Show that $V = \frac{1}{3}\pi\sqrt{16r^2 - r^6}$. **[4]**

a) area $\pi r l = 4\pi$

$\therefore \; l = \frac{4}{r}$

$V = \frac{1}{3}\pi r^2 h$

but $h^2 = l^2 - r^2$

$h = \sqrt{l^2 - r^2}$

$h = \sqrt{\left(\frac{4}{r}\right)^2 - r^2}$

$\therefore \; V = \frac{1}{3} \times \pi \times r^2 \times \sqrt{\frac{16}{r^2} - \frac{r^4}{r^2}}$

$= \frac{1}{3}\pi r^2 \sqrt{\frac{16 - r^4}{r^2}}$

$= \frac{1}{3}\pi r^2 \frac{\sqrt{16 - r^4}}{r}$

$= \frac{1}{3}\pi r \sqrt{16 - r^4}$

$= \frac{1}{3}\pi \sqrt{16r^2 - r^6}$

$(16r^2 - r^6)^{\frac{1}{2}}$

$= (r^2)^{\frac{1}{2}}(16 - r^4)^{\frac{1}{2}}$

(b) In this question you must show detailed reasoning.

It is given that V has a maximum value for a certain value of r.

Find the maximum value of V, giving your answer correct to 3 significant figures. **[5]**

$$V = \frac{\pi}{3}\left(16r^2 - r^6\right)^{\frac{1}{2}}$$

$$\frac{dV}{dr} = \frac{\pi}{3} \times \frac{1}{2}\left(16r^2 - r^6\right)^{-\frac{1}{2}} \times \left(32r - 6r^5\right)$$

$$= \frac{\pi\left(32r - 6r^5\right)}{6\sqrt{16r^2 - r^6}}$$

$\frac{dV}{dr} = 0$ at maximum

$$\therefore \ 32r - 6r^5 = 0$$

$$2r\left(16 - 3r^4\right) = 0$$

$r = 0$ or $3r^4 = 16$

impossible

$$r = \sqrt[4]{\frac{16}{3}}$$

$$\therefore \ \max V = \frac{\pi}{3}\left(16\left(\sqrt[4]{\frac{16}{3}}\right)^2 - \left(\sqrt[4]{\frac{16}{3}}\right)^6\right)^{\frac{1}{2}}$$

$$V = 5.197478 = 5.20 \text{ m}^3 \ (3 \text{ sf})$$

6 Shona makes the following claim.

"n is an **even** positive integer greater than 2 $\Rightarrow 2^n - 1$ is **not** prime"

Prove that Shona's claim is true. **[4]**

if n is even let $n = 2k$

$\therefore\ 2^n - 1 = 2^{2k} - 1$

$= (2^k)^2 - 1$

$= (2^k + 1)(2^k - 1)$

this is product of 2 integers

Can either 2^k+1 or $2^k - 1$ be 1

$2^k + 1 = 1$

$2^k = 0$ impossible

$2^k - 1 = 1$

$2^k = 2$

$k = 1$

can be true for $k = 1$ but n (from question) is greater than 2

$\therefore$ as both integers, but neither is 1, $2^n - 1$ is not prime

7 **In this question you must show detailed reasoning.**

Use the substitution $u = 6x^2 + x$ to solve the equation $36x^4 + 12x^3 + 7x^2 + x - 2 = 0$. **[5]**

$u = 6x^2 + x$

$u^2 = 36x^4 + 12x^3 + x^2$

$(36x^4 + 12x^3 + x^2) + (6x^2 + x) - 2 = 0$

$\uparrow u^2 \qquad \uparrow u$

$u^2 + u - 2 = 0$

$(u + 2)(u - 1)$

$u = -2$ or $u = 1$

if $u = -2$ $\quad -2 = 6x^2 + x$

$0 = 6x^2 + x + 2$

no real roots

if $u = 1$ $\quad 1 = 6x^2 + x$

$0 = 6x^2 + x - 1$

$x = \frac{1}{3}$ or $-\frac{1}{2}$

8 The stem-and-leaf diagram shows the heights, in centimetres, of 17 plants, measured correct to the nearest centimetre.

LQ = 61

median = 65

5	5 7 9 9
6	3 4 5 5 5 9 9
7	4 5 7 9 9
8	
9	9

UQ = 76

Key: 5 | 6 means 56

(a) Find the median and inter-quartile range of these heights. **[3]**

(b) Calculate the mean and standard deviation of these heights. **[2]**

(c) State one advantage of using the median rather than the mean as a measure of average for these heights. **[1]**

a)

median = 65

IQR = UQ − LQ

= 76 − 61

= 15

b) mean and standard deviation

$\bar{x} = 69.0588$

$\sigma = 10.468$

c) less affected by outlier 99

on Classwiz

Menu 6

OPT 1-variable calc

2: y=a+bx

9 **(a)** The masses, in grams, of plums of a certain kind have the distribution N(55, 18).

(i) Find the probability that a plum chosen at random has a mass between 50.0 and 60.0 grams. **[1]**

(ii) The heaviest 5% of plums are classified as extra large.

Find the minimum mass of extra large plums. **[1]**

(iii) The plums are packed in bags, each containing 10 randomly selected plums.

Find the probability that a bag chosen at random has a total mass of less than 530 g. **[4]**

$\mu = 55 \quad \sigma^2 = 18$

(i) Menu 7

2 : Normal CD

Lower 50

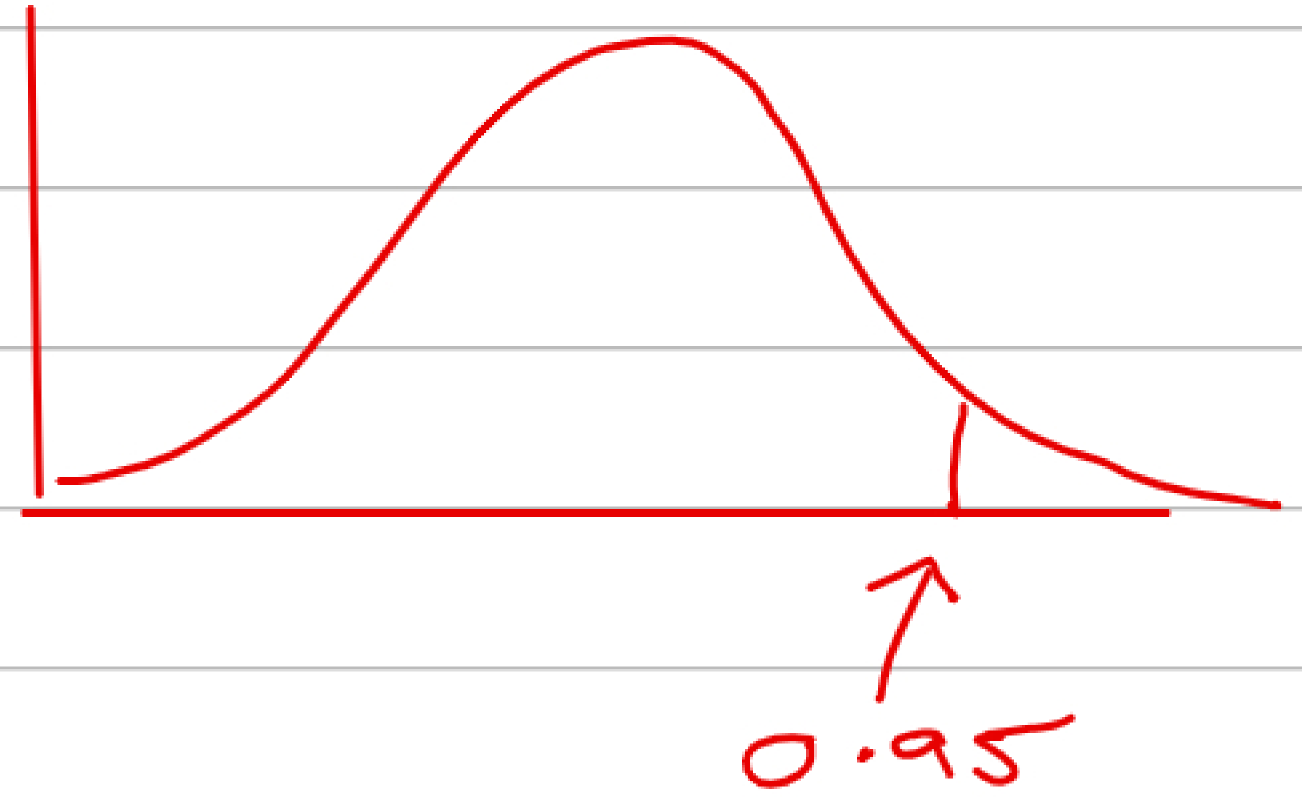
Upper 60

$\sigma = \sqrt{18}$

$\mu = 55$

$p = 0.761$ (3sf)

(ii)



Inverse normal

Area = 0.95

$\sigma = \sqrt{18}$

$\mu = 55$

$p = 61.978 = 62.0$ (3sf)

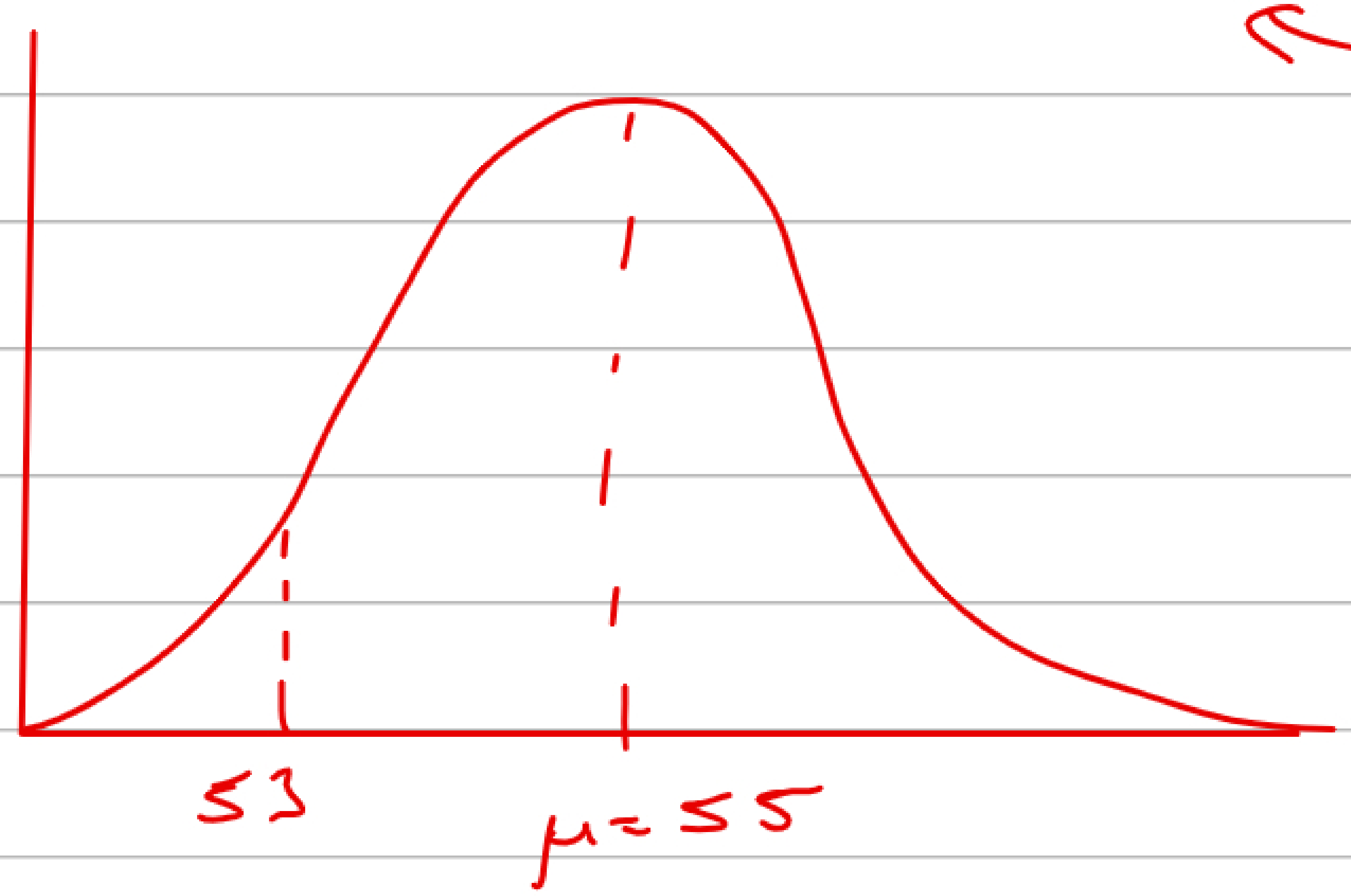
(iii) The plums are packed in bags, each containing 10 randomly selected plums.

Find the probability that a bag chosen at random has a total mass of less than 530 g. **[4]**

10 plums, mass < 530g

means $\bar{X} < \frac{530}{10}$

$\bar{X} < 53$ ← sample mean



when using sample mean

let $\sigma = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{18}}{\sqrt{10}} = 1.3416$

Normal CD

lower $= -1000$

upper $= 53$

$\sigma = 1.3416$

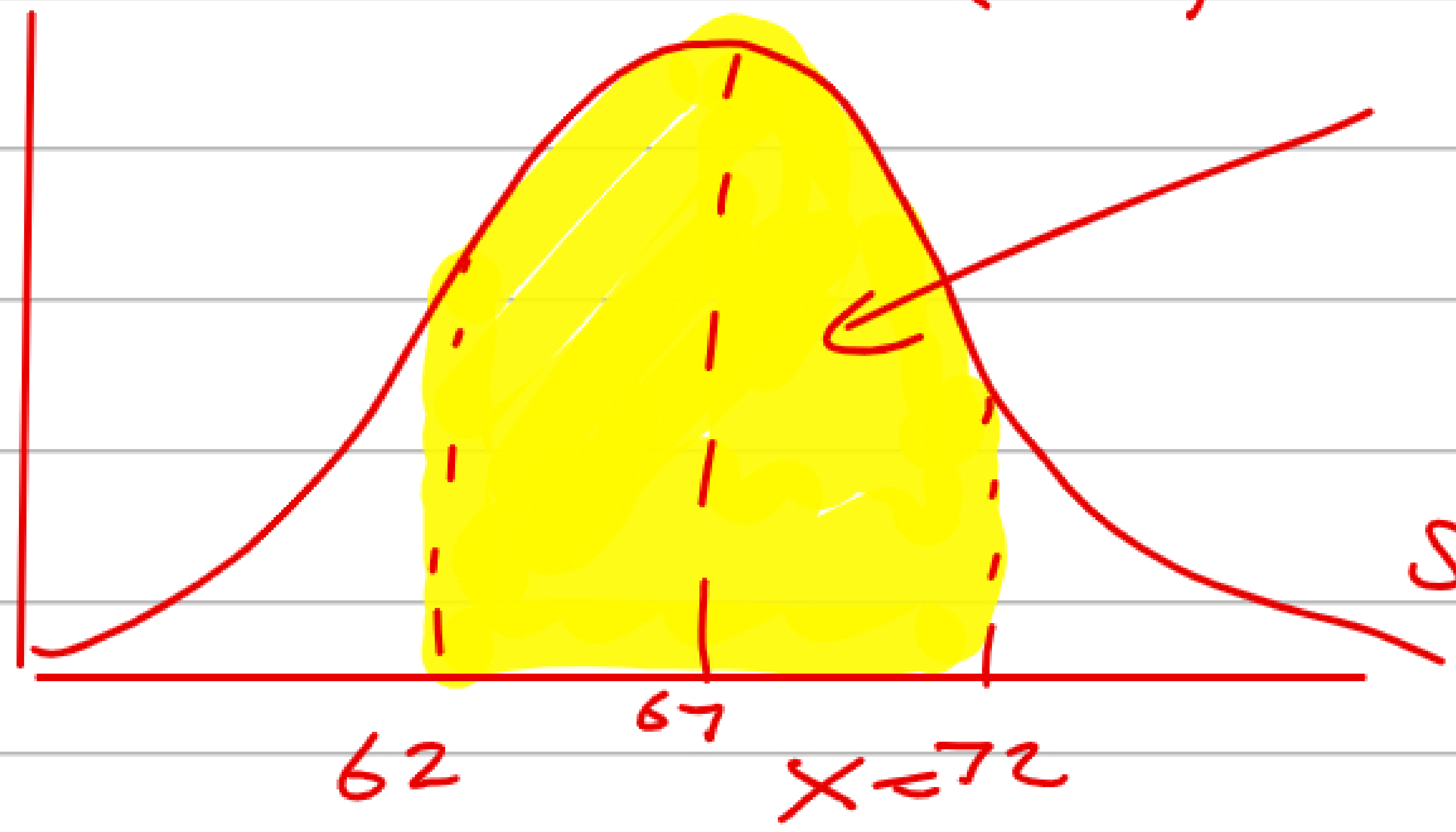
$\mu = 55$

$p = 0.0680$ (3sf)

(b) The masses, in grams, of apples of a certain kind have the distribution $N(67, \sigma^2)$. It is given that half of the apples have masses between 62 g and 72 g.

Determine σ. **[5]**

$N(67, \sigma^2)$

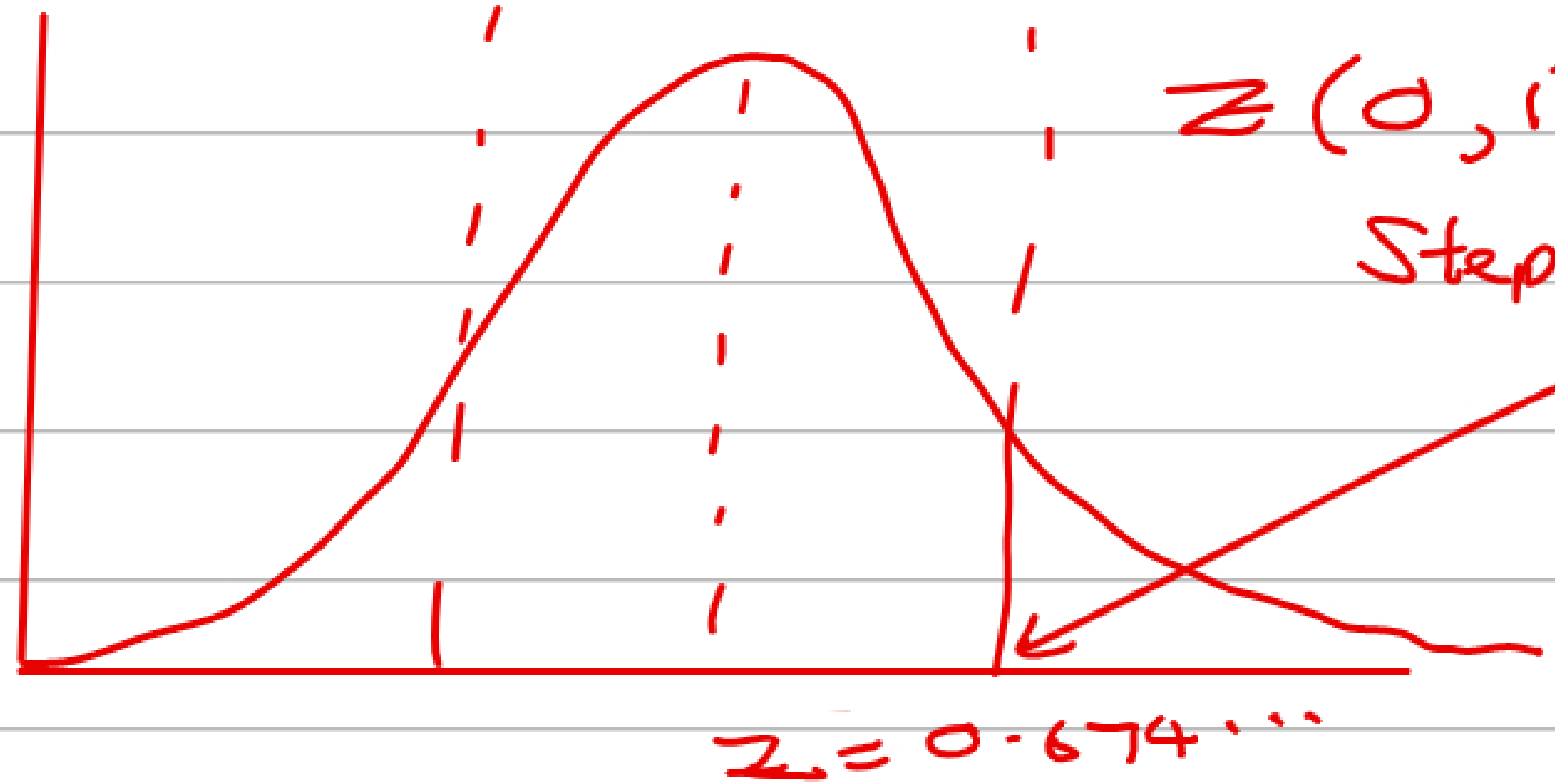


shaded area is 50%

Step 2 $z = \dfrac{x-\mu}{\sigma}$

$\sigma = \dfrac{x-\mu}{z} = \dfrac{72-67}{0.674489579}$

$\sigma = 7.41$ (3 sf)



$Z(0, 1^2)$

Step 1 Inverse normal using $Z(0, 1^2)$

area = 0.75

$\sigma = 1$

$\mu = 0$

$z = 0.674489579$

10 The level, in grams per millilitre, of a pollutant at different locations in a certain river is denoted by the random variable X, where X has the distribution $N(\mu, 0.000\,040\,9)$.

$X \sim N(\mu, 0.0000409)$

σ^2

In the past the value of μ has been 0.0340.

This year the mean level of the pollutant at 50 randomly chosen locations was found to be 0.0325 grams per millilitre.

$n = 50$

sample mean $\bar{x}$

Test, at the 5% significance level, whether the mean level of pollutant has changed. **[7]**

$\div 2 = 2.5\%$ each tail

either direction — 2 tail test

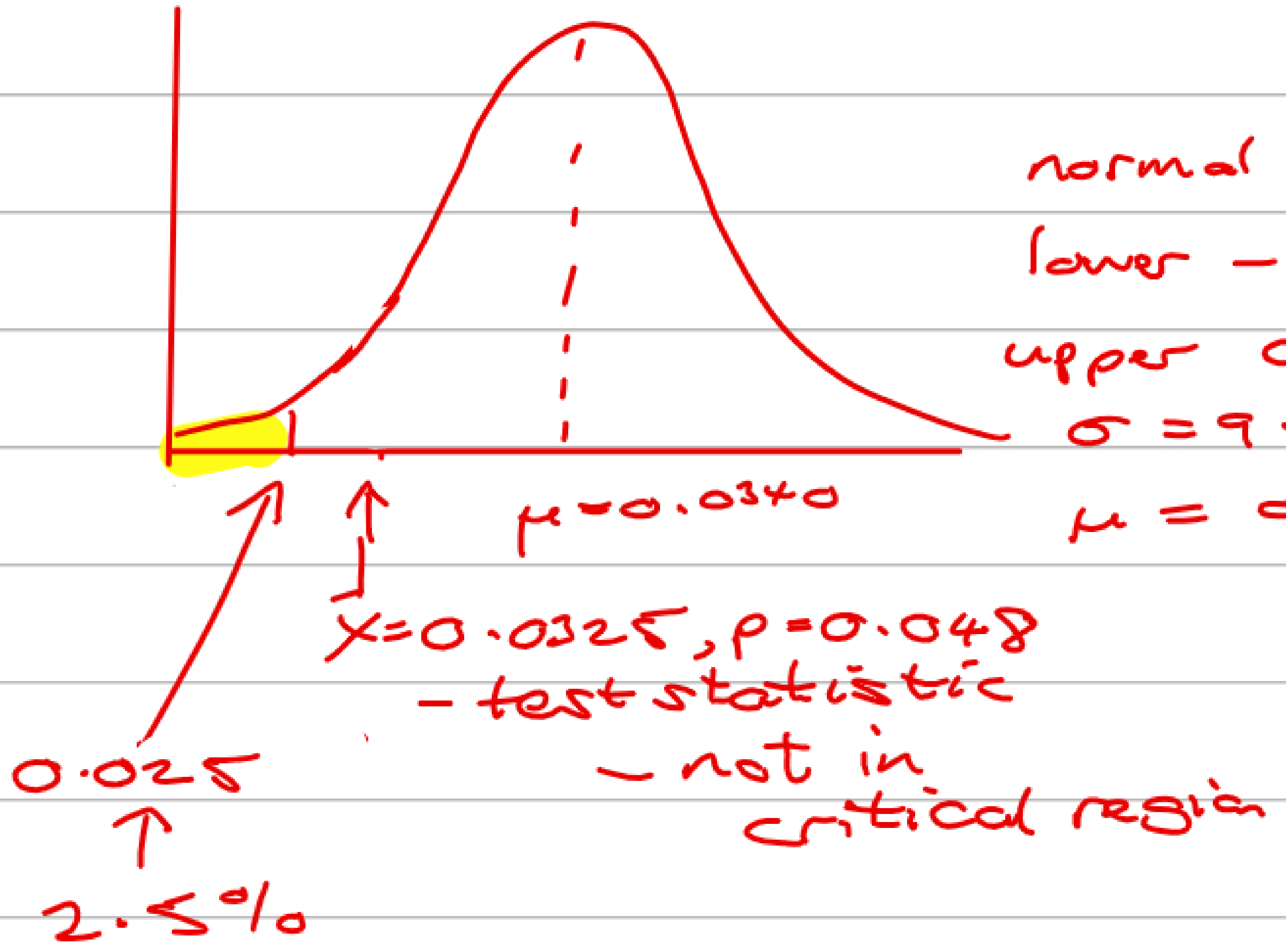
$H_0 : \mu = 0.0340$

$H_0 : \mu \neq 0.0340$

amend σ as sample mean

$$\sigma = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{0.0000409}}{\sqrt{50}}$$

$\sigma = 9.0443 \times 10^{-4}$



normal CD

lower -1000

upper 0.0325

$\sigma = 9.0443 \times 10^{-4}$

$\mu = 0.0340$

$p = 0.048$

2.5% lower tail

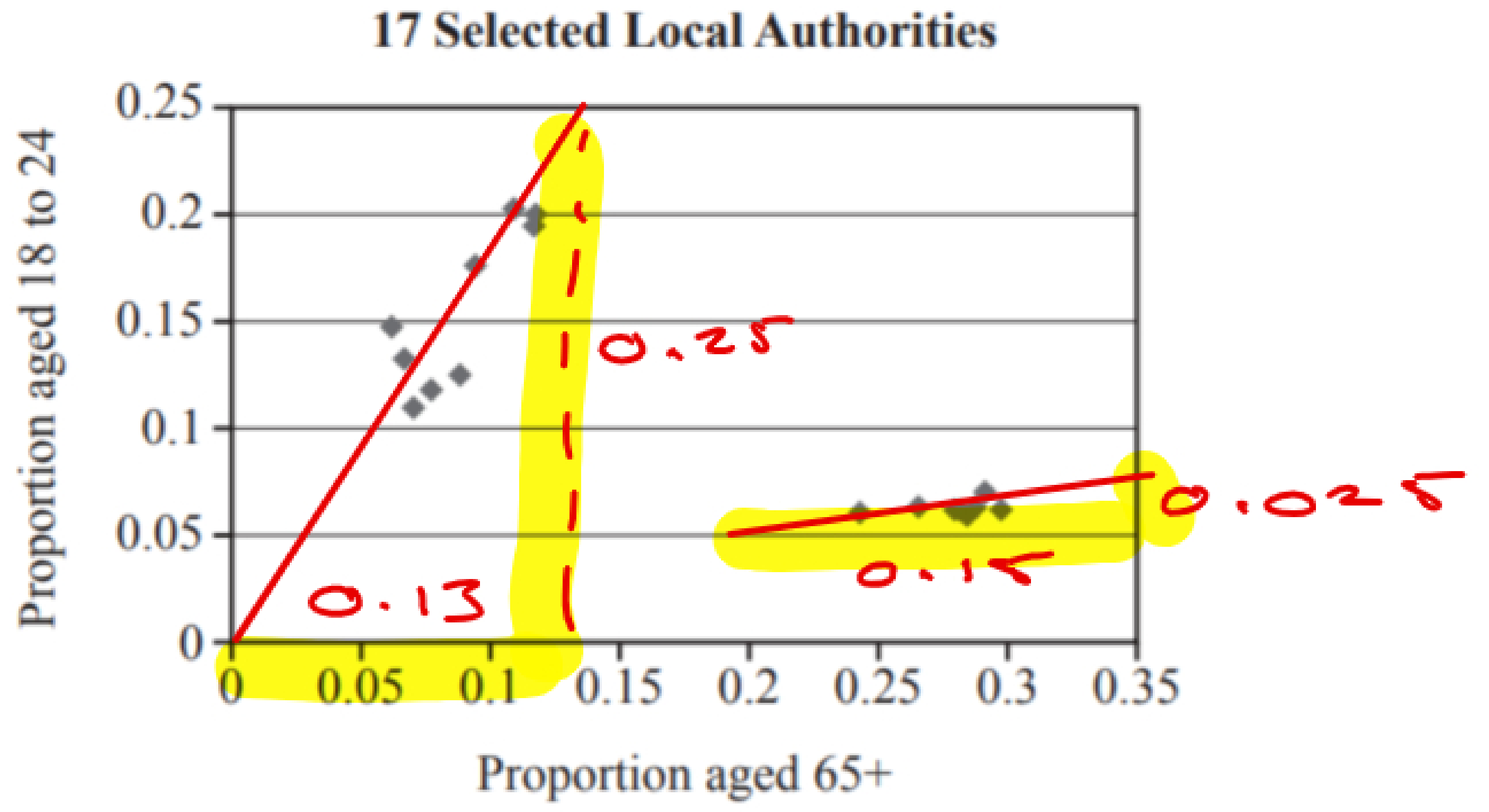
$p = 0.048 > 0.025$

$\therefore$ Ho is not in critical region

accept Ho

Insufficient evidence pollutant level has changed

11 A trainer was asked to give a lecture on population profiles in different Local Authorities (LAs) in the UK. Using data from the 2011 census, he created the following scatter diagram for 17 selected LAs.



$\frac{0.25}{0.13} = 1.92$

$\frac{0.025}{0.15} = 0.16$

$0.16 < k < 1.92$

He selected the 17 LAs using the following method. The proportions of people aged 18 to 24 and aged 65+ in any Local Authority are denoted by P_{young} and P_{senior} respectively. The trainer used a spreadsheet to calculate the value of $k = \frac{P_{\text{young}}}{P_{\text{senior}}}$ for each of the 348 LAs in the UK. He then used specific ranges of values of k to select the 17 LAs.

(a) Estimate the ranges of values of k that he used to select these 17 LAs. **[2]**

(b) Using the 17 LAs the trainer carried out a hypothesis test with the following hypotheses.

H_0: There is no linear correlation in the population between P_{young} and P_{senior}.
H_1: There is negative linear correlation in the population between P_{young} and P_{senior}.

He found that the value of Pearson's product-moment correlation coefficient for the 17 LAs is −0.797, correct to 3 significant figures.

(i) Use the table on page 9 to show that this value is significant at the 1% level. **[2]**

$H_0: \rho = 0$ 1 tail test
$H_1: \rho < 0$

$-0.797 < -0.5577$ (in critical region)

In critical region ∴ reject H_0
Accept H_1, there is evidence of correlation at 1% level

Critical values of Pearson's product-moment correlation coefficient

1-tail test	5%	2.5%	1%	0.5%
2-tail test	10%	5%	2%	1%
n				
1	-	-	-	-
2	-	-	-	-
3	0.9877	0.9969	0.9995	0.9999
4	0.9000	0.9500	0.9800	0.9900
5	0.8054	0.8783	0.9343	0.9587
6	0.7293	0.8114	0.8822	0.9172
7	0.6694	0.7545	0.8329	0.8745
8	0.6215	0.7067	0.7887	0.8343
9	0.5822	0.6664	0.7498	0.7977
10	0.5494	0.6319	0.7155	0.7646
11	0.5214	0.6021	0.6851	0.7348
12	0.4973	0.5760	0.6581	0.7079
13	0.4762	0.5529	0.6339	0.6835
14	0.4575	0.5324	0.6120	0.6614
15	0.4409	0.5140	0.5923	0.6411
16	0.4259	0.4973	0.5742	0.6226
17	0.4124	0.4821	0.5577	0.6055
18	0.4000	0.4683	0.5425	0.5897
19	0.3887	0.4555	0.5285	0.5751
20	0.3783	0.4438	0.5155	0.5614
21	0.3687	0.4329	0.5034	0.5487
22	0.3598	0.4227	0.4921	0.5368
23	0.3515	0.4132	0.4815	0.5256
24	0.3438	0.4044	0.4716	0.5151
25	0.3365	0.3961	0.4622	0.5052
26	0.3297	0.3882	0.4534	0.4958
27	0.3233	0.3809	0.4451	0.4869
28	0.3172	0.3739	0.4372	0.4785
29	0.3115	0.3673	0.4297	0.4705
30	0.3061	0.3610	0.4226	0.4629

(ii) Use the diagram to comment on the reliability of this conclusion. **[2]**

There are 2 clusters (or groups) which show correlation.
The 2 clusters have different correlation values.

(c) Describe one outstanding feature of the population in the areas represented by the points in the bottom right hand corner of the diagram. **[1]**

(d) The trainer's audience included representatives from several universities.

Suggest a reason why the diagram might be of particular interest to these people. **[1]**

c) High proportion of 65+

d) The top left points contain high proportion of 18-24s, so there may be LA's where there is a university, where they can recruit.

12 A random variable X has probability distribution defined as follows.

$$P(X=x)=\begin{cases} kx & x=1,2,3,4,5, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Show that $P(X=3)=0.2$. **[3]**

(b) Show in a table the values of X and their probabilities. **[2]**

(c) Two independent values of X are chosen, and their total T is found.

(i) Find $P(T=7)$. **[3]**

(ii) Given that $T=7$, determine the probability that one of the values of X is 2. **[4]**

a) $P(X=3) = 3k$

$P(X = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5)$

$= k + 2k + 3k + 4k + 5k = 15k$

$15k = 1 \Rightarrow k = \frac{1}{15}$

$P(X=3) = 3k = 3 \times \frac{1}{15} = \frac{1}{5} = 0.2$ (as required)

(b) Show in a table the values of X and their probabilities. **[2]**

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(c) Two independent values of X are chosen, and their total T is found.

(i) Find $P(T = 7)$. **[3]**

2 and 5 $= \frac{2}{15} \times \frac{5}{15} = \frac{10}{225}$

5 and 2 $= \frac{2}{15} \times \frac{5}{15} = \frac{10}{225}$

3 and 4 $= \frac{3}{15} \times \frac{4}{15} = \frac{12}{225}$

4 and 3 $= \frac{4}{15} \times \frac{3}{15} = \frac{12}{225}$

$P(X=7) = \frac{44}{225}$

(ii) Given that $T = 7$, determine the probability that one of the values of X is 2. **[4]**

2 and 5   5 and 2

$$\frac{\frac{10}{225} + \frac{10}{225}}{\frac{44}{225}} = \frac{5}{11}$$

P(T=7)

13 It is known that 26% of adults in the UK use a certain app. A researcher selects a random sample of 5000 adults in the UK. The random variable X is defined as the number of adults in the sample who use the app.

Given that $P(X < n) < 0.025$, calculate the largest possible value of n. **[5]**

Approximate Binomial with Normal for large n

$n = 5000$

$p = 0.26$

approximate with normal if $np > 5$ and $n(1-p) > 5$

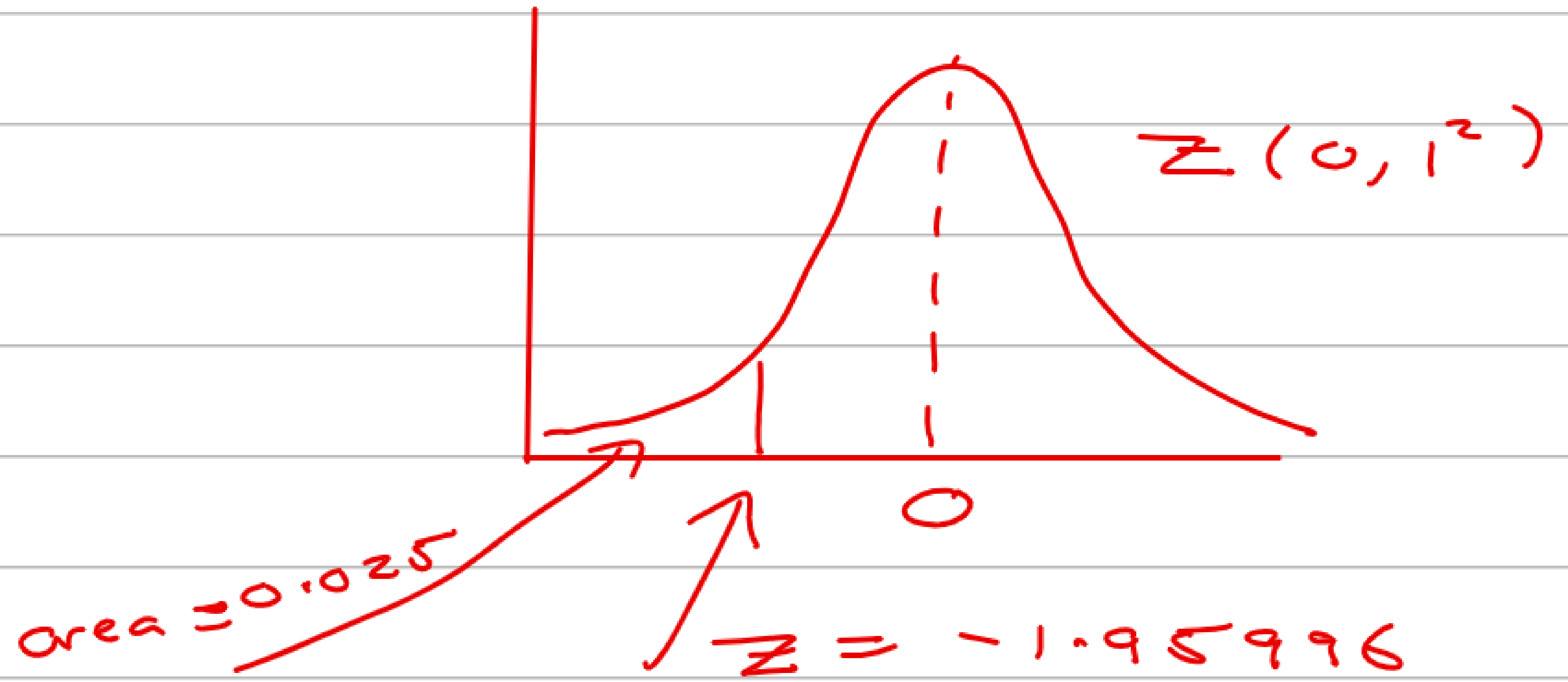
$np = 5000 \times 0.26 = 1300$

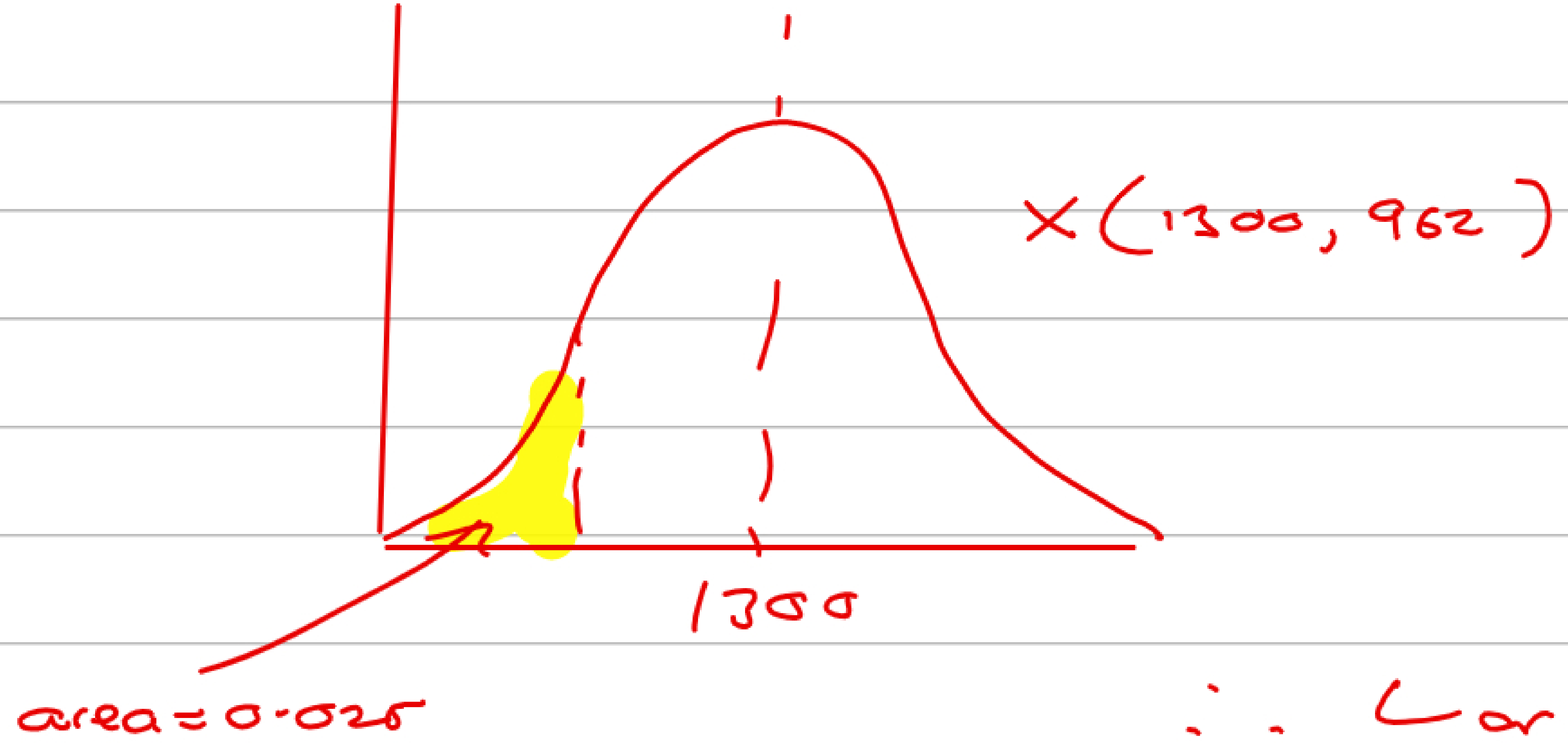
$n(1-p) = 5000 \times 0.74 = 3700$

$\mu = np = 1300$

$\sigma = \sqrt{np(1-p)} = \sqrt{1300 \times 0.74} = \sqrt{962}$

$X \sim N(1300, 962)$





Inverse normal using Z

Area = 0.025

$\sigma = 1$

$\mu = 0$

$Z = -1.95996$

$$Z = \frac{X - \mu}{\sigma}$$

$$X = Z \times \sigma + \mu$$

$$X = -1.95996 \times \sqrt{96^2} + 1300$$

$$X = 1239.2$$

$\therefore$ Largest $n = 1239$