OCR
Oxford Cambridge and RSA

A Level Mathematics A

H240/01 Pure Mathematics

Wednesday 6 June 2018 – Morning

Time allowed: 2 hours

YORKSHIRE
MATHS TUTOR

Solutions

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets [ ].
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

1 The points A and B have coordinates (1, 5) and (4, 17) respectively. Find the equation of the straight line which passes through the point (2, 8) and is perpendicular to AB. Give your answer in the form $ax + by = c$, where a, b and c are constants. [4]

$$m = \frac{17-5}{4-1} = \frac{12}{3} = 4$$

perpendicular gradient $= -\frac{1}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - 8 = -\frac{1}{4}(x-2)$$

$$4y - 32 = -x + 2$$
$$x + 4y = 34$$

2 (i) Use the trapezium rule, with four strips each of width 0.5, to estimate the value of

$$\int_0^2 e^{x^2}\,dx$$

giving your answer correct to 3 significant figures. [3]

(ii) Explain how the trapezium rule could be used to obtain a more accurate estimate. [1]

(i)

x	0	0.5	1	1.5	2
$y = e^{x^2}$	1	1.2840	2.7183	9.4877	54.5982 (4dp)

$$\text{Area} = \frac{1}{2} \times 0.5\left(1 + 54.5982 + 2\left(1.2840 + 2.7183 + 9.4877\right)\right)$$

$$= 20.64455$$

$$= 20.6 \quad (3\text{ sf})$$

(ii) Use more trapezia, narrow the width of each trapezium

3 In this question you must show detailed reasoning.

Find the two real roots of the equation $x^4 - 5 = 4x^2$. Give the roots in an exact form. **[4]**

$x^4 - 4x^2 - 5 = 0$

$(x^2 - 5)(x^2 + 1) = 0$

$x^2 - 5 = 0$ or $x^2 + 1 = 0$

$x^2 = -1$

impossible

$x^2 = 5$

$x = \sqrt{5}$ or $-\sqrt{5}$

4 Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n.

If n is even, let $n = 2m$

$$(2m)^3 + 3 \times 2m - 1 = 8m^3 + 6m - 1$$
$$= 2(4m^3 + 3m) - 1$$

as $2(4m^3+3m)$ is even, then $2(4m^3+3m)-1$ is odd

If n is odd, let $n = 2m+1$

$$(2m+1)^3 + 3(2m+1) - 1$$
$$= (2m+1)(4m^2+4m+1) + 6m + 3 - 1$$
$$= 8m^3 + 8m^2 + 2m + 4m^2 + 4m + 1 + 6m + 3 - 1$$
$$= 8m^3 + 12m^2 + 12m + 3$$
$$= 2(4m^3 + 6m^2 + 6m) + 3$$

as $2(4m^3+6m^2+6m)$ is even, then $2(4m^2+6m^2+6m)+3$ is odd

$\therefore$ $n^3 + 3n - 1$ is odd for all integers n

5 The equation of a circle is $x^2+y^2+6x-2y-10=0$.

(i) Find the centre and radius of the circle. **[3]**

(ii) Find the coordinates of any points where the line $y=2x-3$ meets the circle $x^2+y^2+6x-2y-10=0$. **[4]**

(iii) State what can be deduced from the answer to part **(ii)** about the line $y=2x-3$ and the circle $x^2+y^2+6x-2y-10=0$. **[1]**

(i) $x^2+6x+y^2-2y-10=0$

$(x+3)^2-3^2+(y-1)^2-1^2-10=0$

$(x+3)^2+(y-1)^2=10+1+9$

$(x+3)^2+(y-1)^2=20$

Centre $(-3,1)$, radius $\sqrt{20}$

(ii) Find the coordinates of any points where the line $y = 2x - 3$ meets the circle $x^2 + y^2 + 6x - 2y - 10 = 0$. **[4]**

(iii) State what can be deduced from the answer to part **(ii)** about the line $y = 2x - 3$ and the circle $x^2 + y^2 + 6x - 2y - 10 = 0$. **[1]**

(ii) $x^2 + (2x-3)^2 + 6x - 2(2x-3) - 10 = 0$

$x^2 + 4x^2 - 12x + 9 + 6x - 4x + 6 - 10 = 0$

$5x^2 - 10x + 5 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1$, $y = 2 \times 1 - 3 = -1$

$(1, -1)$

(iii) The line only touches circle at $(1, -1)$ so is a tangent

6 The cubic polynomial $f(x)$ is defined by $f(x) = 2x^3 - 7x^2 + 2x + 3$.

(i) Given that $(x-3)$ is a factor of $f(x)$, express $f(x)$ in a fully factorised form. **[3]**

(ii) Sketch the graph of $y = f(x)$, indicating the coordinates of any points of intersection with the axes. **[2]**

(iii) Solve the inequality $f(x) < 0$, giving your answer in set notation. **[2]**

(iv) The graph of $y = f(x)$ is transformed by a stretch parallel to the x-axis, scale factor $\frac{1}{2}$. Find the equation of the transformed graph. **[2]**

(i)

$$
\begin{array}{r|l}
 & 2x^2 - x - 1 \\
x-3 & 2x^3 - 7x^2 + 2x + 3 \\
 & -\ (2x^3 - 6x^2) \\
 & -x^2 + 2x \\
 & -\ (-x^2 + 3x) \\
 & -x + 3 \\
 & -x + 3 \\
 & 0
\end{array}
$$

$2x^2 - x - 1$

$= (2x+1)(x-1)$

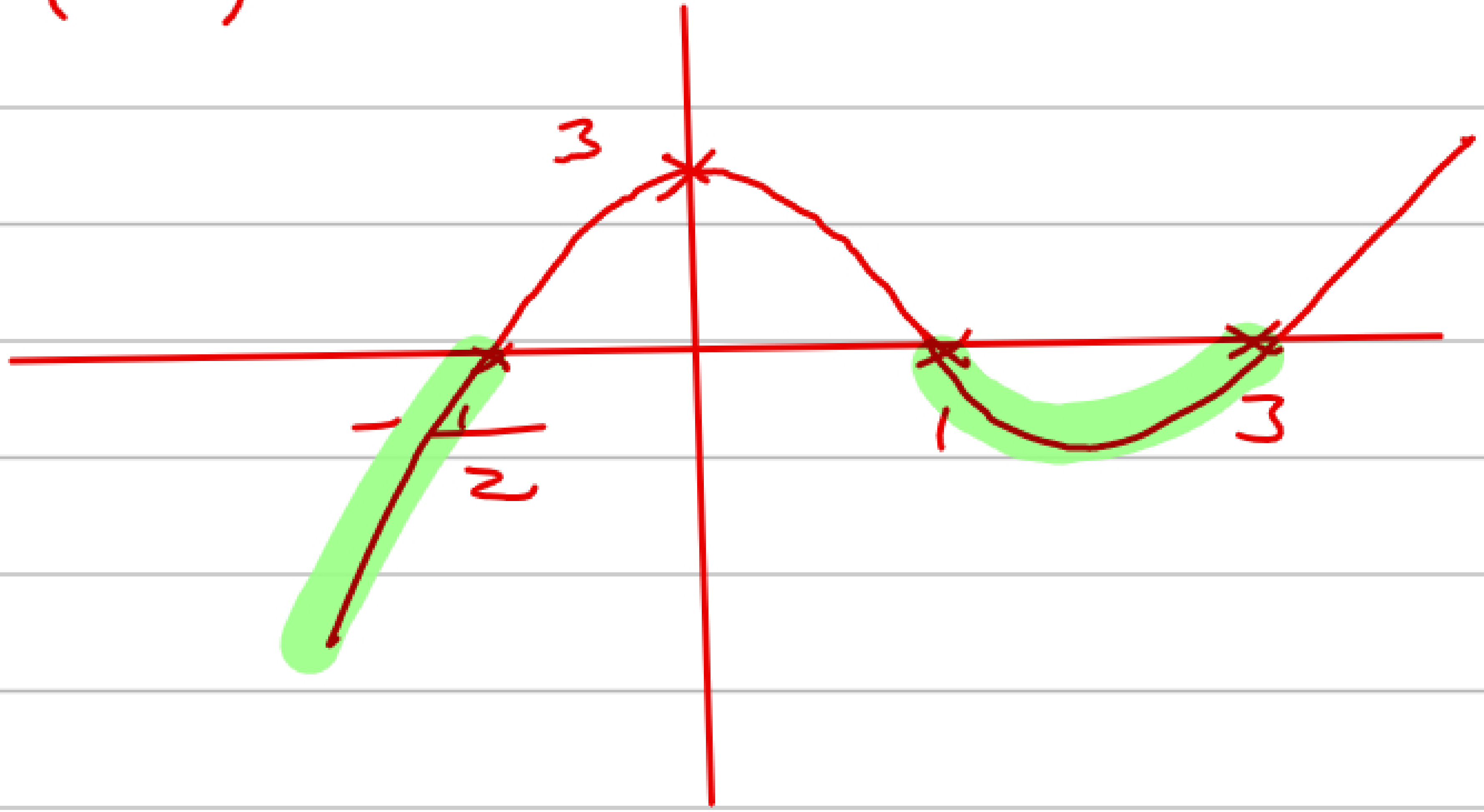
$f(x) = (x-3)(2x+1)(x-1)$

(ii) Sketch the graph of $y = \mathrm{f}(x)$, indicating the coordinates of any points of intersection with the axes. **[2]**

(iii) Solve the inequality $\mathrm{f}(x) < 0$, giving your answer in set notation. **[2]**

(iv) The graph of $y = \mathrm{f}(x)$ is transformed by a stretch parallel to the x-axis, scale factor $\frac{1}{2}$. Find the equation of the transformed graph. **[2]**

(ii)



$f(x) = (x-3)(2x+1)(x-1)$

at $x = 0$, $y = 3$

(iii) $f(x) < 0$ on parts of graph shaded green

$\{x : x < -\frac{1}{2}\} \cup \{x : 1 < x < 3\}$

in set notation

7 Chris runs half marathons, and is following a training programme to improve his times. His time for his first half marathon is 150 minutes. His time for his second half marathon is 147 minutes. Chris believes that his times can be modelled by a geometric progression.

(i) Chris sets himself a target of completing a half marathon in less than 120 minutes. Show that this model predicts that Chris will achieve his target on his thirteenth half marathon. **[4]**

(ii) After twelve months Chris has spent a total of 2974 minutes, to the nearest minute, running half marathons. Use this model to find how many half marathons he has run. **[3]**

(iii) Give two reasons why this model may not be appropriate when predicting the time for a half marathon. **[2]**

nth term is ar^{n-1}

(i) 1st 2nd

150 147

a ar

$r = \frac{147}{150}$

$a = 150$

solve $ar^{n-1} < 120$

$150 \times \left(\frac{147}{150}\right)^{n-1} < 120$

$\left(\frac{147}{150}\right)^{n-1} < \frac{120}{150}$

$(n-1)\ln(0.98) < \ln(0.8)$

$n\ln(0.98) - \ln(0.98) <$

$n\ln(0.98) < \ln(0.8)$

$n > \frac{\ln(0.8) + \ln(0.98)}{\ln(0.98)}$

$n > 12.045$

$\therefore n = 13$ is thirteenth

(ii) After twelve months Chris has spent a total of 2974 minutes, to the nearest minute, running half marathons. Use this model to find how many half marathons he has run. **[3]**

(iii) Give two reasons why this model may not be appropriate when predicting the time for a half marathon. **[2]**

$$S_{12} = \frac{a(1-r^n)}{1-r}$$

$$2974 = \frac{150(1-0.98^n)}{1-0.98}$$

$$2974 \times (0.02) = 150(1-0.98^n)$$

$$\frac{2974 \times 0.02}{150} = 1 - 0.98^n$$

$$= \frac{1487}{3750}$$

$$0.98^n = 1 - \frac{1487}{3750}$$

$$n \ln(0.98) = \ln\left(1 - \frac{1487}{3750}\right)$$

$$n = \frac{\ln\left(1 - \frac{1487}{3750}\right)}{\ln(0.98)} = 24.999$$

$= 25$ marathons

(iii) Give two reasons why this model may not be appropriate when predicting the time for a half marathon. **[2]**

Could be variations in conditions between marathons

Model assumes his times will continue to improve at the same rate

8 **(i)** Find the first three terms in the expansion of $(4-x)^{-\frac{1}{2}}$ in ascending powers of x. **[4]**

(ii) The expansion of $\dfrac{a+bx}{\sqrt{4-x}}$ is $16-x\ \dots$. Find the values of the constants a and b. **[3]**

(i) $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1 + \frac{(-\frac{1}{2})}{1}\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1\times 2}\left(-\frac{x}{4}\right)^2 + \dots\right)$

$= \frac{1}{2}\left(1 + \frac{1}{8}x + \frac{3}{128}x^2 + \dots\right)$

$= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \dots$

(ii) $(a+bx)\left(\frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \dots\right)$

$= \frac{1}{2}a + \frac{1}{16}ax + \frac{3}{256}ax^2$

$+ \frac{1}{2}bx + \frac{1}{16}bx^2 + \dots$

Coefficients

constant $\Rightarrow$ $\frac{1}{2}a = 16$ $\quad a = 32$

x $\Rightarrow$ $\frac{1}{16}\times 32 + \frac{1}{2}b = -1$

$2 + \frac{1}{2}b = -1$

$\frac{1}{2}b = -3$

$b = -6$

$a = 32$

$b = -6$

9 The function f is defined for all real values of x as $f(x) = c + 8x - x^2$, where c is a constant.

(i) Given that the range of f is $f(x) \leqslant 19$, find the value of c. **[3]**

(ii) Given instead that $ff(2) = 8$, find the possible values of c. **[4]**

(i)

$f(x) = -x^2 + 8x + c$

$f'(x) = -2x + 8$

$0 = -2x + 8$

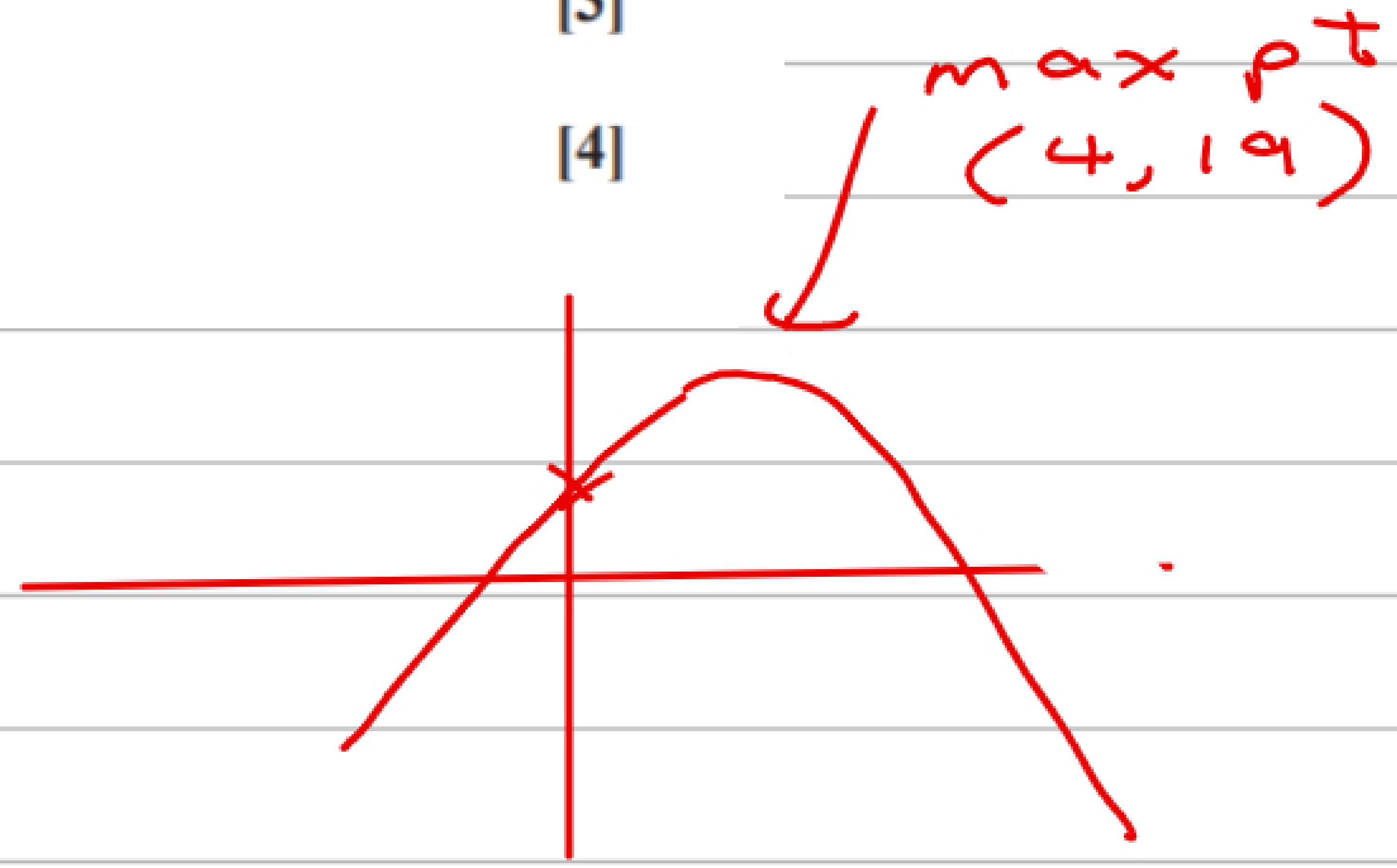
$x = 4$

at $x = 4$ $f(x) = -16 + 32 + c$

$\therefore 19 = -16 + 32 + c$

$19 + 16 - 32 = c$

$c = 3$



(ii) Given instead that ff(2) = 8, find the possible values of c.

$$f(x) = -x^2 + 8x + c$$

$$f(2) = -4 + 16 + c = 12 + c$$

$$f(12+c) = -(12+c)^2 + 8(12+c) + c$$
$$8 = -(144 + 24c + c^2) + 96 + 8c + c$$
$$8 = -144 - 24c - c^2 + 96 + 8c + c$$
$$8 = -c^2 - 15c - 48$$
$$c^2 + 15c + 56 = 0$$
$$(c+7)(c+8) = 0$$
$$c = -7 \quad \text{or} \quad c = -8$$

10 A curve has parametric equations $x = t + \frac{2}{t}$ and $y = t - \frac{2}{t}$, for $t \neq 0$.

(i) Find $\frac{dy}{dx}$ in terms of t, giving your answer in its simplest form. **[4]**

(ii) Explain why the curve has no stationary points. **[2]**

(iii) By considering $x + y$, or otherwise, find a cartesian equation of the curve, giving your answer in a form not involving fractions or brackets. **[4]**

(i)

$$x = t + 2t^{-1} \qquad y = t - 2t^{-1}$$

$$\frac{dx}{dt} = 1 - 2t^{-2} \qquad \frac{dy}{dt} = 1 + 2t^{-2}$$

$$\frac{dy}{dx} = \frac{1 + \frac{2}{t^2}}{1 - \frac{2}{t^2}} = \frac{\frac{t^2+2}{t^2}}{\frac{t^2-2}{t^2}} = \frac{t^2+2}{t^2-2}$$

(ii) $\frac{dy}{dx} = 0$ at stationary points

$\therefore t^2 + 2 = 0$

$t^2 = -2$

impossible to get square root of -2

(iii) By considering $x + y$, or otherwise, find a cartesian equation of the curve, giving your answer in a form not involving fractions or brackets. **[4]**

1 2

$x = t + \frac{2}{t}$ and $y = t - \frac{2}{t}$, for $t \neq 0$.

$$x + y = t + \frac{2}{t} + t - \frac{2}{t}$$

$$x + y = 2t$$

$$\frac{x+y}{2} = t$$

sub in 2

$$y = \frac{x+y}{2} - \frac{2}{\frac{x+y}{2}}$$

$$y = \frac{x+y}{2} - \frac{4}{x+y}$$

× by $2(x+y)$

$$2y(x+y) = \frac{\cancel{2}(x+y)(x+y)}{\cancel{2}} - \frac{4}{\cancel{(x+y)}} \times 2\cancel{(x+y)}$$

$$\cancel{2xy} + 2y^2 = x^2 + \cancel{2xy} + y^2 - 8$$

$$2y^2 = x^2 + y^2 - 8$$

$$8 = x^2 - y^2$$

11 In a science experiment a substance is decaying exponentially. Its mass, M grams, at time t minutes is given by $M = 300e^{-0.05t}$.

(i) Find the time taken for the mass to decrease to half of its original value. [3]

A second substance is also decaying exponentially. Initially its mass was 400 grams and, after 10 minutes, its mass was 320 grams.

(ii) Find the time at which both substances are decaying at the same rate. [8]

(i) $M = 300e^{-0.05t}$

at $t = 0$, $M = 300 \times e^{-0} = 300g$

$150 = 300e^{-0.05t}$

$\frac{150}{300} = e^{-0.05t}$

$\ln\left(\frac{1}{2}\right) = -0.05t$

$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.05} = 13.9$ minutes

A second substance is also decaying exponentially. Initially its mass was 400 grams and, after 10 minutes, its mass was 320 grams.

(ii) Find the time at which both substances are decaying at the same rate. $\frac{dM}{dt}$ **[8]**

(ii) $M_2 = M_0 e^{kt}$ at $t = 0$ mass $= 400$

$M_2 = 400 e^{kt}$

at $t = 10$, $M_2 = 320$

$320 = 400 e^{10k}$

$\frac{320}{400} = e^{10k}$ $= 0.8$

$\ln\left(\frac{320}{400}\right) = 10k$

$k = \frac{1}{10} \ln(0.8)$

$M_2 = 400 e^{\frac{1}{10}\ln(0.8)t}$

$\frac{dM_2}{dt} = \frac{1}{10}\ln(0.8) \times 400 e^{\frac{1}{10}\ln(0.8)t}$

$\frac{dM_2}{dt} = -8.9257 e^{-0.0223t}$

$$M_1 = 300 e^{-0.05t}$$

$$\frac{dM_1}{dt} = 300 \times (-0.05) e^{-0.05t}$$

$$\frac{dM_1}{dt} = -15 e^{-0.05t}$$

when same rate $\frac{dM_1}{dt} = \frac{dM_2}{dt}$

$$-15 e^{-0.05t} = -8.9257 e^{-0.0223t}$$

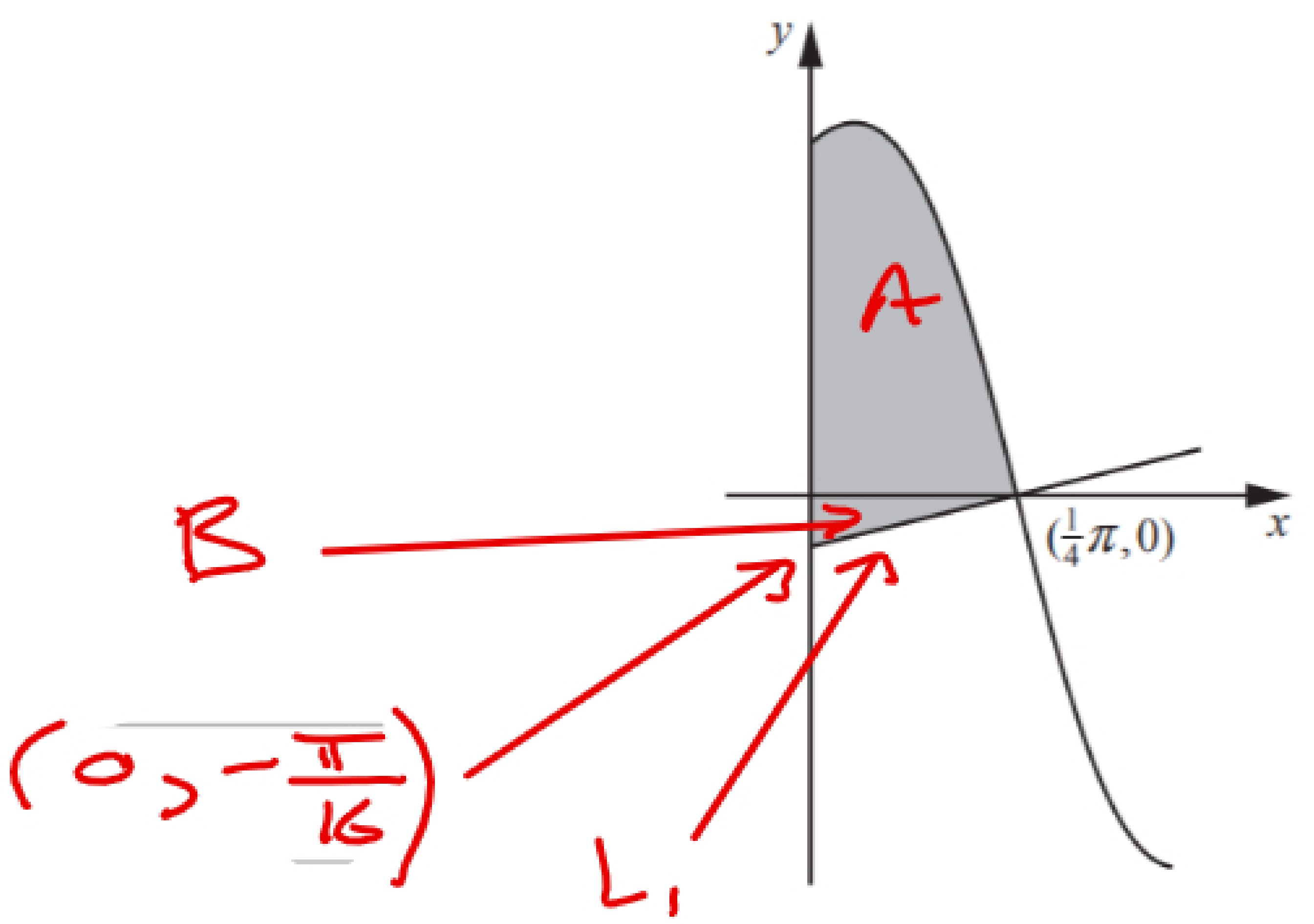
$$\frac{e^{-0.05t}}{e^{-0.0223t}} = \frac{-8.9257}{-15}$$

$$e^{-0.0277t} = \frac{-8.9257}{-15}$$

$$-0.0277t = \ln\left(\frac{-8.9257}{-15}\right)$$

$$t = \frac{\ln\left(\frac{-8.9257}{-15}\right)}{-0.0277} = 18.7 \text{ minutes (3 sf)}$$

12 **In this question you must show detailed reasoning.**



The diagram shows the curve $y = \dfrac{4\cos 2x}{3 - \sin 2x}$, for $x \geqslant 0$, and the normal to the curve at the point $(\frac{1}{4}\pi, 0)$. Show that the exact area of the shaded region enclosed by the curve, the normal to the curve and the y-axis is $\ln\frac{9}{4} + \frac{1}{128}\pi^2$. **[10]**

A area under curve

$$\int_0^{\frac{1}{4}\pi} \frac{4\cos 2x}{3 - \sin 2x}\,dx$$

Consider $y = \ln|3 - \sin 2x|$

$$\frac{dy}{dx} = \frac{-2\cos 2x}{3 - \sin 2x}$$

So $I = -2\ln|3 - \sin 2x|$

$$\left[-2\ln|3 - \sin 2x|\right]_0^{\frac{1}{4}\pi}$$

$$= \left(-2\ln\left|3 - \sin\frac{\pi}{2}\right|\right) - \left(-2\ln 3\right)$$

$$= -2\ln 2 + 2\ln 3$$

$$= 2(\ln 3 - \ln 2)$$

$$= 2\ln\frac{3}{2} = \ln\left(\frac{3}{2}\right)^2$$

$$= \ln\frac{9}{4} \quad 1$$

The diagram shows the curve $y = \dfrac{4\cos 2x}{3 - \sin 2x}$, for $x \geqslant 0$, and the normal to the curve at the point $(\frac{1}{4}\pi, 0)$. Show that the exact area of the shaded region enclosed by the curve, the normal to the curve and the y-axis is $\ln\frac{9}{4} + \frac{1}{128}\pi^2$. **[10]**

Find gradient of curve at $x = \frac{\pi}{4}$ Quotient rule

$u = 4\cos 2x \qquad v = 3 - \sin 2x$

$\dfrac{du}{dx} = -8\sin 2x \qquad \dfrac{dv}{dx} = -2\cos 2x$

$$\frac{dy}{dx} = \frac{-8\sin 2x(3 - \sin 2x) + 8(\cos 2x)^2}{(3 - \sin 2x)^2}$$

at $x = \frac{\pi}{4}$ $$\frac{dy}{dx} = \frac{-8\sin\frac{\pi}{2}(3 - \sin\frac{\pi}{2}) + 8(\cos\frac{\pi}{2})^2}{(3 - \sin\frac{\pi}{2})^2}$$

$$\frac{dy}{dx} = -4$$

$\therefore$ gradient of normal $= \frac{1}{4}$ line l_1 $(\frac{\pi}{4}, 0)$ $m = \frac{\pi}{4}$

$y - 0 = \frac{1}{4}(x - \frac{\pi}{4})$

$y = \frac{1}{4}x - \frac{\pi}{16}$

Area of triangle B

$$= \frac{1}{2} \times \frac{\pi}{16} \times \frac{\pi}{4} = \frac{\pi^2}{128} \quad (2)$$

$\therefore$ Total area $= (1) + (2)$

$$= \ln\frac{9}{4} + \frac{\pi^2}{128}$$

(as required)

13 A scientist is attempting to model the number of insects, N, present in a colony at time t weeks. When $t = 0$ there are 400 insects and when $t = 1$ there are 440 insects.

(i) A scientist assumes that the rate of increase of the number of insects is inversely proportional to the number of insects present at time t.

(a) Write down a differential equation to model this situation. **[1]**

(b) Solve this differential equation to find N in terms of t. **[4]**

(ii) In a revised model it is assumed that $\frac{dN}{dt} = \frac{N^2}{3988e^{0.2t}}$. Solve this differential equation to find N in terms of t. **[6]**

(iii) Compare the long-term behaviour of the two models. **[2]**

(i) a) $\frac{dN}{dt} = k \frac{1}{N}$

b) $\int N \, dN = \int k \, dt$

$\frac{1}{2} N^2 = kt + c$

$t=0, N=400$ $\quad \frac{1}{2} \times 400^2 = c \Rightarrow c = 80000$

$\frac{1}{2} N^2 = kt + 80000$

$t=1, N=440$ $\quad \frac{1}{2} \times 440^2 = k + 80000$

$k = 96800 - 80000 = 16800$

$$\frac{1}{2}N^2 = 16800t + 80000$$
$$N^2 = 33600t + 160000$$
$$N = \sqrt{33600t + 160000}$$

(ii) In a revised model it is assumed that $\frac{dN}{dt} = \frac{N^2}{3988e^{0.2t}}$. Solve this differential equation to find N in terms of t. **[6]**

(iii) Compare the long-term behaviour of the two models. **[2]**

$$\int N^{-2}\,dN = \int \frac{1}{3988} e^{-0.2t}\,dt$$
$$-N^{-1} = \frac{1}{3988 \times -0.2} \times e^{-0.2t} + c$$
$$-\frac{1}{N} = -\frac{5}{3988} e^{-0.2t} + c$$

$t = 0, N = 400$
$$-\frac{1}{400} = -\frac{5}{3988} + c \Rightarrow c = \frac{5}{3988} - \frac{1}{400} = -\frac{497}{398800}$$

$$-\frac{1}{N} = -\frac{5}{3988} e^{-0.2t} - \frac{497}{398800}$$

$\times$ through by $-N$

$$1 = N\left(\frac{5}{3988} e^{-0.2t} + \frac{497}{398800}\right)$$

$$1 = N\left(\frac{500 e^{-0.2t} + 497}{398800}\right)$$

$$\frac{398800}{500 e^{-0.2t} + 497} = N \quad (2)$$

(iii) Model (1) $N = \sqrt{33600t + 160000}$ will increase to ∞

Model (2) $N = \dfrac{398800}{500 e^{-0.25t} + 497}$ as $t \to$ large

$N \to$ limit ≈ 802

$(t < 1 \times 10^{80})$