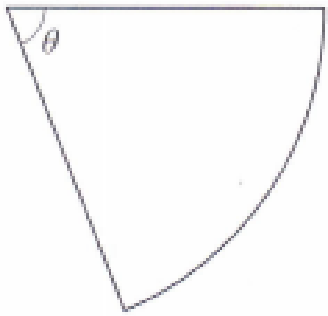
3 The diagram below shows a sector of a circle.



Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 4^2 \times 0.8$

$= 6.4 \text{cm}^2$

The radius of the circle is 4 cm and $\theta = 0.8$ radians.

Find the area of the sector.

Circle your answer.

[1 mark]

1.28 cm^2 $\quad$ 3.2 cm^2 $\quad$ 6.4 cm^2 $\quad$ 12.8 cm^2

1 $\mathrm{f}(x) = \arcsin x$

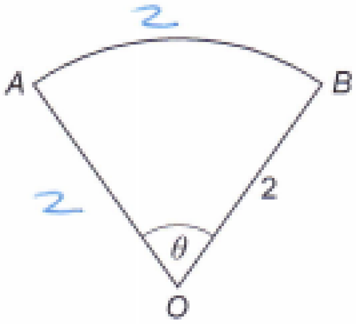
State the maximum possible domain of f

Tick (✓) **one** box.

[1 mark]

- $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$ ✓
- $\left\{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$ ☐
- $\{x \in \mathbb{R} : -\pi \leq x \leq \pi\}$ ☐
- $\{x \in \mathbb{R} : -90 \leq x \leq 90\}$ ☐

3 The diagram shows a sector OAB of a circle with centre O and radius 2



The angle AOB is θ radians and the perimeter of the sector is 6

Find the value of θ

$AB = 2$

Circle your answer.

$r \times \theta = 2 \Rightarrow \theta = 1$

[1 mark]

1 $\sqrt{3}$ 2 3

Turn over for the next question

4 Using small angle approximations, show that for small, non-zero, values of x

$$\frac{x \tan 5x}{\cos 4x - 1} \approx A$$

where A is a constant to be determined.

[4 marks]

small angles $\tan\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$\frac{x \times 5x}{1 - \frac{(4x)^2}{2} - 1} \approx -\frac{5x^2}{16x^2} \times 2$$

$$\approx -\frac{10}{16} \approx -\frac{5}{8}$$

Do not write outside the box

8 (a) Prove the identity $\dfrac{\sin 2x}{1+\tan^2 x} \equiv 2\sin x\cos^3 x$

$\sin 2x = 2\sin x\cos x$
$\sec^2 x = 1+\tan^2 x$

[3 marks]

$$\frac{2\sin x\cos x}{1+(\sec^2 x-1)} = \frac{2\sin x\cos x}{\sec^2 x}$$

$= 2\sin x\cos x \times \cos^2 x$

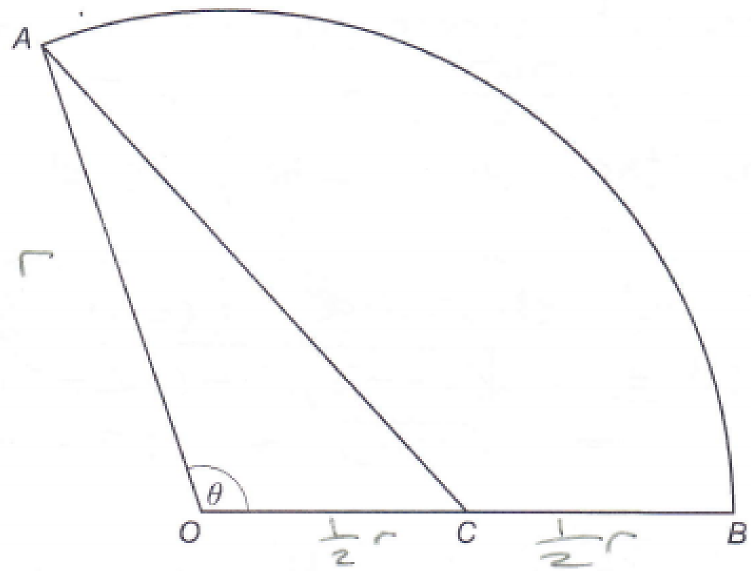
$= 2\sin x\cos^3 x$ (as required)

Do not write outside the box

8 The diagram shows a sector of a circle OAB.

C is the midpoint of OB.

Angle AOB is θ radians.



8 (a) Given that the area of the triangle OAC is equal to one quarter of the area of the sector OAB, show that $\theta = 2\sin\theta$

[4 marks]

Area sector $OAB = \frac{1}{2}r^2\theta$

Area $\triangle OAC = \frac{1}{4} \times \frac{1}{2} \times r^2\theta$

$= \frac{1}{8}r^2\theta$

Area of $\triangle = \frac{1}{2}ab\sin C$

$\therefore \frac{1}{2} \times r \times \frac{1}{2}r \times \sin\theta = \frac{1}{8}r^2\theta$

$\frac{1}{4}\cancel{r^2}\sin\theta = \frac{1}{8}\cancel{r^2}\theta$

$\frac{8}{4}\sin\theta = \theta$

$2\sin\theta = \theta$ (as required)

12 (a) Show that the equation

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

can be written in the form

$$a\operatorname{cosec}^2 x + b\operatorname{cosec} x + c = 0$$

[2 marks]

$2(\operatorname{cosec}^2 x - 1) + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$

$2\operatorname{cosec}^2 x - 2 + 2\operatorname{cosec}^2 x - 1 - 4\operatorname{cosec} x = 0$

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

$\dfrac{\sin^2 x}{\sin^2 x} + \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{1}{\sin^2 x}$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

12 (b) Hence, given x is obtuse and

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

find the exact value of $\tan x$

Fully justify your answer.

[5 marks]

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

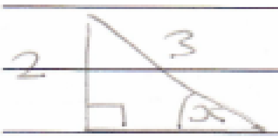
$(2\operatorname{cosec} x + 1)(2\operatorname{cosec} x - 3) = 0$

$\operatorname{cosec} x = -\frac{1}{2}$ $\qquad$ $\operatorname{cosec} x = \frac{3}{2}$

$\frac{1}{\sin x} = -\frac{1}{2}$ $\qquad$ $\frac{1}{\sin x} = \frac{3}{2}$

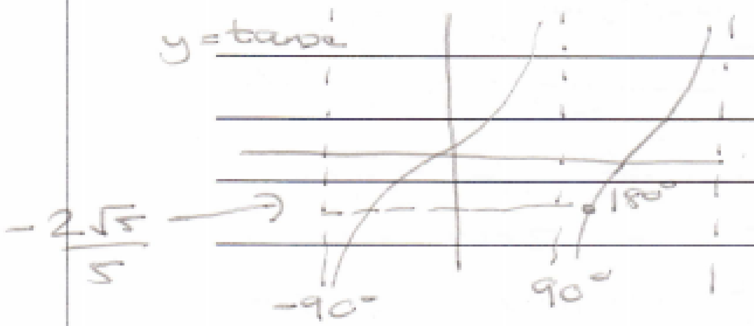
$\sin x = -2$ $\qquad$ $\sin x = \frac{2}{3}$

impossible



$\sqrt{5}$ $\qquad$ $\sqrt{3^2-2^2} = \sqrt{5}$

$\tan x = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$



But for x to be obtuse, from diagram

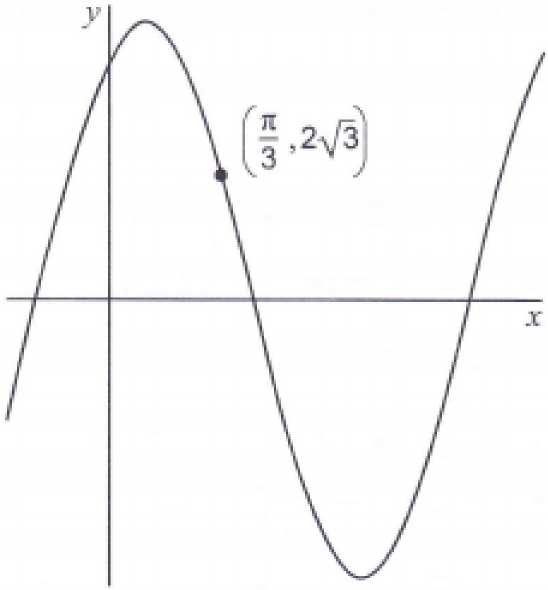
$\tan x = -\frac{2\sqrt{5}}{5}$

6 A curve has equation

$$y = a\sin x + b\cos x$$

where a and b are constants.

The maximum value of y is 4 and the curve passes through the point $\left(\frac{\pi}{3}, 2\sqrt{3}\right)$ as shown in the diagram.



Find the exact values of a and b.

[6 marks]

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$a = R\cos\alpha$

$b = R\sin\alpha$

$\therefore \quad \tan\alpha = \frac{b}{a}$ Max value $= 4$

$R = \sqrt{a^2+b^2} = 4$ (1)

at $x = \frac{\pi}{3}$, $y = 2\sqrt{3}$

$\therefore \quad 2\sqrt{3} = \frac{\sqrt{3}}{2}a + \frac{1}{2}b$

$4\sqrt{3} = \sqrt{3}a + b$

$4\sqrt{3} - \sqrt{3}a = b$ (2)

in (1) $\sqrt{a^2 + (4\sqrt{3} - \sqrt{3}a)^2} = 4$

$$\sqrt{a^2 + 48 - 24a + 3a^2} = 4$$

$$\sqrt{4a^2 - 24a + 48} = 4$$

squaring

$$4a^2 - 24a + 48 = 16$$

$$4a^2 - 24a + 32 = 0$$

$$a^2 - 6a + 8 = 0$$

$$(a - 4)(- 2) = 0$$

$a = 4$ or $a = 2$

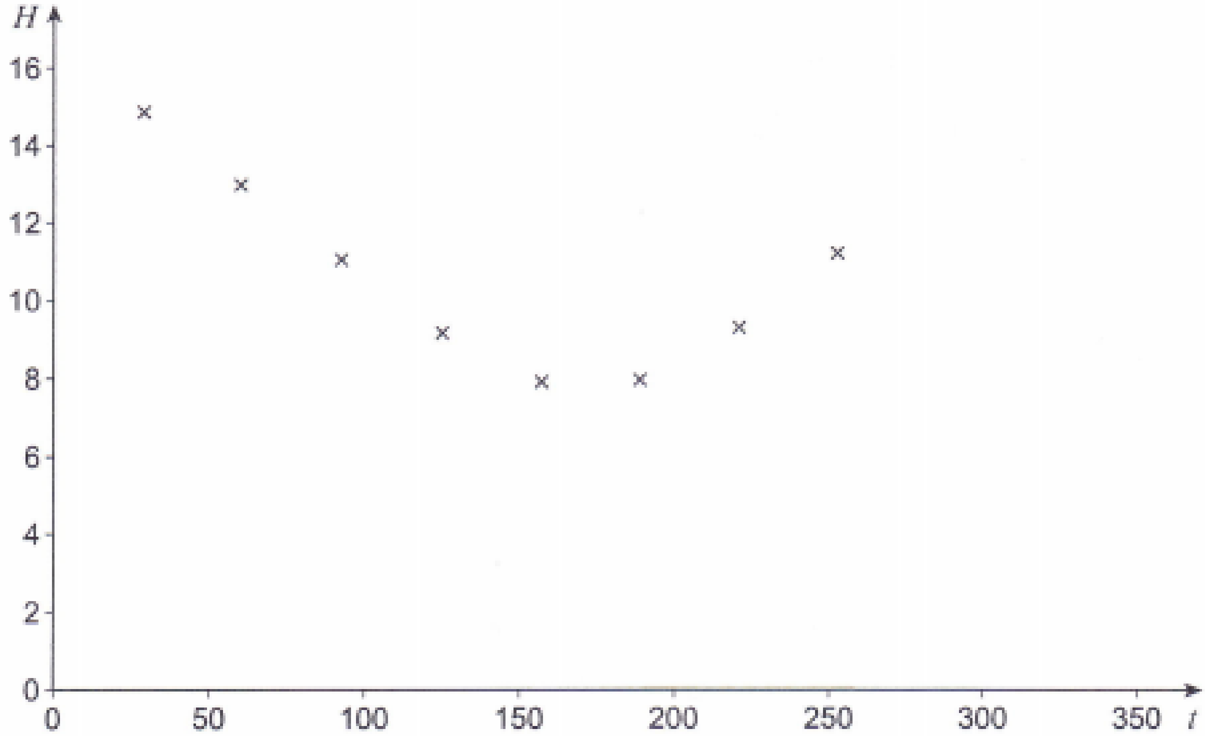
$b = 0$ or $b = 2\sqrt{3}$

8 Mike, an amateur astronomer who lives in the South of England, wants to know how the number of hours of darkness changes through the year.

On various days between February and September he records the length of time, H hours, of darkness along with t, the number of days after 1 January.

His results are shown in **Figure 1** below.

Figure 1



Mike models this data using the equation

$$H = 3.87\sin\left(\frac{2\pi(t + 101.75)}{365}\right) + 11.7$$

8 (a) Find the minimum number of hours of darkness predicted by Mike's model. Give your answer to the nearest minute.

[2 marks]

min val $\sin(\quad) = -1$

min $H = 11.7 - 3.87$

$= 7.83$ hours

$0.83 \times 60 = 49.8$ minutes

7 hours 50 minutes

8 (b) Find the maximum number of consecutive days where the number of hours of darkness predicted by Mike's model exceeds 14

[3 marks]

Find times when $= 14$

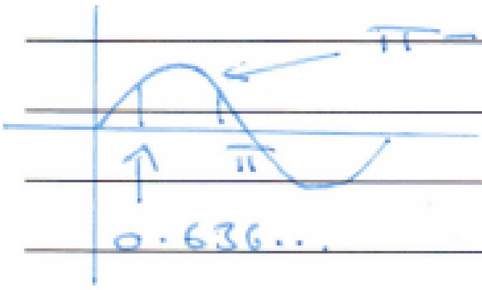
$$3.87\sin\left(\frac{2\pi(t+101.75)}{365}\right)+11.7=14$$

$$\sin\left(\frac{2\pi(t+101.75)}{365}\right)=\frac{14-11.7}{3.87}$$

$$\sin\left(\frac{2\pi(t+101.75)}{365}\right)=\frac{230}{387}$$

$$\sin^{-1}\left(\frac{230}{387}\right)=0.636413947$$

$$\pi-0.636\ldots=2.505178712$$



$$\frac{2\pi(t+101.75)}{365}=0.636413947$$
or 2.505178712

$$t=0.636413947\times\frac{365}{2\pi}-101.75$$
$$=-64.7797\ldots \quad (1)$$

or $$t=2.505178712\times\frac{365}{2\pi}-101.75$$
$$=43.779725 \quad (2)$$

Question 8 continues on the next page

Difference $= (2)-(1) = 108.559425$

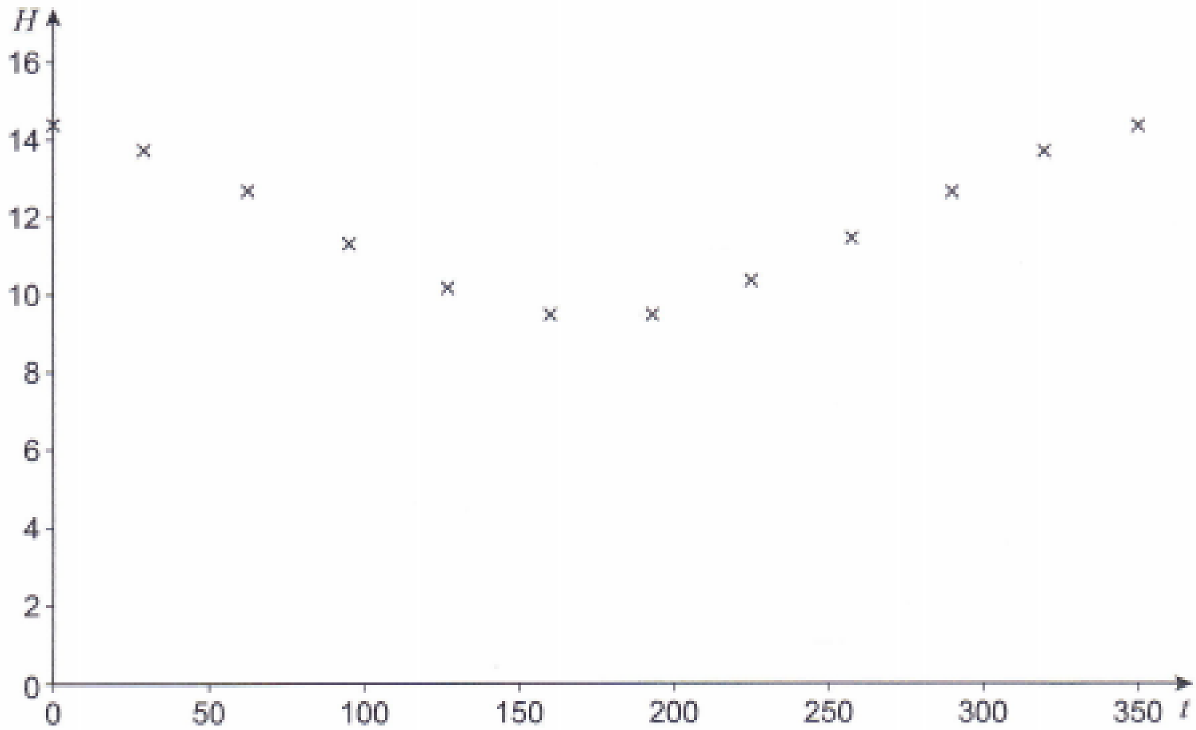
$= 109$ days

(rounded to nearest whole number)

8 (c) Mike's friend Sofia, who lives in Spain, also records the number of hours of darkness on various days throughout the year.

Her results are shown in **Figure 2** below.

Figure 2



Sofia attempts to model her data by refining Mike's model.

She decides to increase the 3.87 value, leaving everything else unchanged.

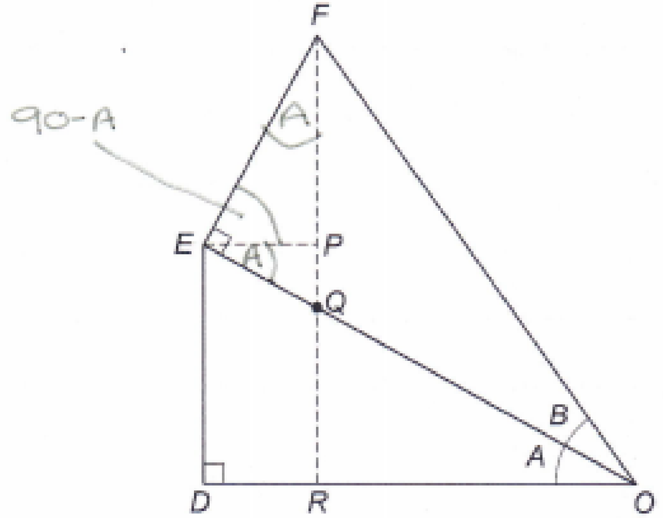
Explain whether Sofia's refinement is appropriate.

[2 marks]

Increasing the 3.87 would increase the amplitude of the curve (vertical stretch) Looking at her curve, it has a smaller amplitude than Mike's, so increasing 3.87 would not be appropriate

14 Some students are trying to prove an identity for $\sin(A+B)$.

They start by drawing two right-angled triangles ODE and OEF, as shown.



The students' incomplete proof continues,

Let angle $DOE = A$ and angle $EOF = B$.

In triangle OFR,

Line 1 $\quad \sin(A+B) = \dfrac{RF}{OF}$

Line 2 $\quad = \dfrac{RP+PF}{OF}$

Line 3 $\quad = \dfrac{DE}{OF} + \dfrac{PF}{OF}$ since $DE = RP$

Line 4 $\quad = \dfrac{DE}{\dots\dots} \times \dfrac{\dots\dots}{OF} + \dfrac{PF}{EF} \times \dfrac{EF}{OF}$

Line 5 $\quad = \dots\dots\dots\dots\dots + \cos A \sin B$

14 (a) Explain why $\dfrac{PF}{EF} \times \dfrac{EF}{OF}$ in Line 4 leads to $\cos A \sin B$ in Line 5

[2 marks]

$\angle PEQ = A$ (alternate)

$\therefore \angle FEP = 90 - A \Rightarrow \angle EFP = A$ (angle sum in triangle)

$\therefore \cos A = \dfrac{PF}{EF}$ ($\triangle EFP$)

and $\sin B = \dfrac{EF}{OF}$ (using $\triangle FEO$)

14 (b) Complete Line 4 and Line 5 to prove the identity

Line 4 $= \dfrac{DE}{EO} \times \dfrac{EO}{OF} + \dfrac{PF}{EF} \times \dfrac{EF}{OF}$

Line 5 $=$ sin A cos B $+ \cos A \sin B$

[1 mark]

14 (c) Explain why the argument used in part **(a)** only proves the identity when A and B are acute angles.

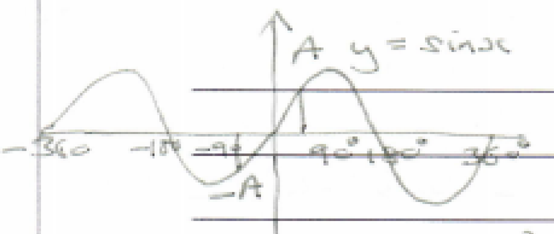
[1 mark]

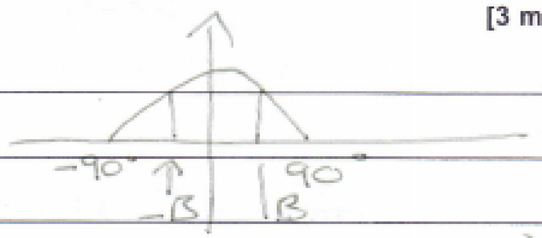
Proof is based on right angled triangles, which rules out obtuse angles

14 (d) Another student claims that by replacing B with $-B$ in the identity for $\sin(A+B)$ it is possible to find an identity for $\sin(A-B)$.

Assuming the identity for $\sin(A+B)$ is correct for all values of A and B, prove a similar result for $\sin(A-B)$.

[3 marks]





$\sin A = -\sin(-A)$ $\cos B = \cos(-B)$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

replacing B with −B gives

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

$= \sin A \cos B - \cos A \sin B$