YORKSHIRE MATHS TUTOR

1 The first three terms, in ascending powers of x, of the binomial expansion of $(9+2x)^{\frac{1}{2}}$ are given by

$$(9+2x)^{\frac{1}{2}} \approx a + \frac{x}{3} - \frac{x^2}{54}$$

where a is a constant.

1 (a) State the range of values of x for which this expansion is valid.

Circle your answer.

[1 mark]

$|x| < \frac{2}{9}$ $\quad$ $|x| < \frac{2}{3}$ $\quad$ $|x| < 1$ $\quad$ $|x| < \frac{9}{2}$

1 (b) Find the value of a.

Circle your answer.

[1 mark]

1 $\quad$ 2 $\quad$ 3 $\quad$ 9

$$(9+2x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{2}{9}x\right)^{\frac{1}{2}}$$

$$9^{\frac{1}{2}} = 3$$

7 Consecutive terms of a sequence are related by

$$u_{n+1} = 3 - (u_n)^2$$

7 (a) In the case that $u_1 = 2$

7 (a) (i) Find u_3

[2 marks]

$u_1 = 2$

$u_2 = -1$

$u_3 = 2$

7 (a) (ii) Find u_{50}

[1 mark]

$u_4 = -1$

all even occurences will be -1

$u_{50} = -1$

7 (b) State a different value for u_1 which gives the same value for u_{50} as found in part **(a)(ii)**.

[1 mark]

$3 - (u_n)^2 = -1$

$4 = (u_n)^2$

$u_1 = -2$

9 An arithmetic sequence has first term a and common difference d.

The sum of the first 36 terms of the sequence is equal to the square of the sum of the first 6 terms.

9 (a) Show that $4a + 70d = 4a^2 + 20ad + 25d^2$

[4 marks]

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{36} = 18(2a + 35d)$

$S_6 = 3(2a + 5d)$

$\therefore \quad 18(2a + 35d) = 9(2a + 5d)^2$

$36a + 630d = 9(4a^2 + 20ad + 25d^2)$

$36a + 630d = 36a^2 + 180ad + 225d^2$

$\div$ through by 9 gives

$4a + 70d = 4a^2 + 20ad + 25d^2$

(as required)

Do not write outside the box

9 (b) Given that the sixth term of the sequence is 25, find the smallest possible value of a.

[5 marks]

6^{th} term $= a + 5d = 25$

$a = 25 - 5d$ (1)

$4(25-5d) + 70d = 4(25-5d)^2 + 20d(25-5d)$

$+ 25d^2$

$100 - 20d + 70d = 4(625 - 250d + 25d^2)$

$+ 500d - 100d^2 + 25d^2$

$100 - 20d + 70d = 2500 - 1000d + \cancel{100d^2}$

$+ 500d - \cancel{100d^2} + 25d^2$

$0 = 25d^2 - 550d + 2400$

$0 = d^2 - 22d + 96$

$0 = (d - 16)(d - 6)$

$d = 16$ or $d = 6$

in (1) $a = 25 - 5 \times 16 = -55$

or $a = 25 - 5 \times 6 = -5$

$a = -55$ is smallest

8 $P(n) = \sum_{k=0}^{n} k^3 - \sum_{k=0}^{n-1} k^3$ where n is a positive integer.

8 (a) Find P(3) and P(10)

[2 marks]

when $n = 3$ $P(3) = 3^3 = 27$

$n = 10$ $P(10) = 10^3 = 1000$

8 (b) Solve the equation $P(n) = 1.25 \times 10^8$

[2 marks]

$n^3 = 1.25 \times 10^8$

$n = \sqrt[3]{1.25 \times 10^8}$

$n = 500$

3 Find the coefficient of x^2 in the binomial expansion of $\left(2x - \frac{3}{x}\right)^8$

[3 marks]

$^8C_0(2x)^8\left(-\frac{3}{x}\right)^0 + {}^8C_1(2x)^7\left(-\frac{3}{x}\right)^1$

$+ {}^8C_2(2x)^6\left(-\frac{3}{x}\right)^2$

$+ {}^8C_3(2x)^5\left(-\frac{3}{x}\right)^3 + \dots$

Only term to produce x^2 is

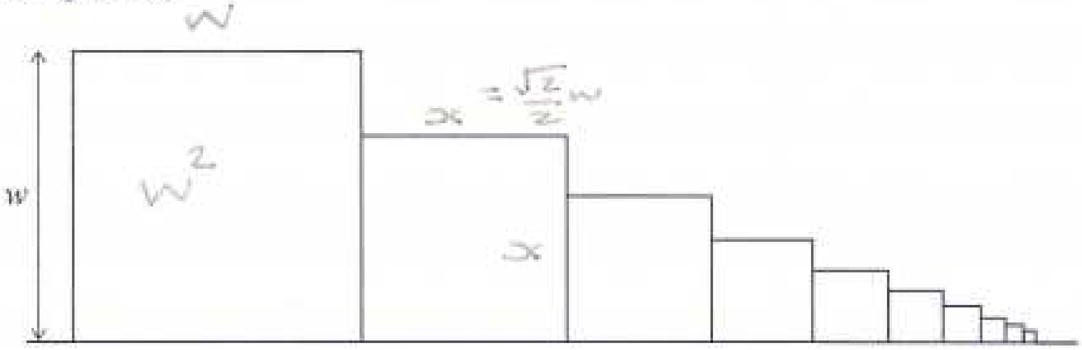
$^8C_3(2x)^5\left(-\frac{3}{x}\right)^3$

$= 56 \times 32x^5 \times -\frac{27}{x^3}$

$= -48384x^2$

coefficient of x^2 is -48384

9 Helen is creating a mosaic pattern by placing square tiles next to each other along a straight line.



The area of each tile is half the area of the previous tile, and the sides of the largest tile have length w centimetres.

9 (a) Find, in terms of w, the length of the sides of the second largest tile.

[1 mark]

$\frac{1}{2}w^2 = x^2$

$x = \sqrt{\frac{w^2}{2}} = \frac{w}{\sqrt{2}} = \frac{\sqrt{2}}{2}w$

9 (b) Assume the tiles are in contact with adjacent tiles, but do not overlap.

Show that, no matter how many tiles are in the pattern, the total length of the series of tiles will be less than $3.5w$.

[4 marks]

Geometric progression

$a = w$

$r = \frac{\sqrt{2}}{2}$

$S_\infty = \frac{a}{1-r} = \frac{w}{1-\frac{\sqrt{2}}{2}} =$

$= (2+\sqrt{2})w$

$\approx 3.4142w$

where $3.4142 < 3.5$

outside the box

6 (a) Find the first three terms, in ascending powers of x, of the binomial expansion of $\frac{1}{\sqrt{4+x}}$

[3 marks]

$(4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1 + \frac{(-\frac{1}{2})}{1}\left(\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1\times 2}\left(\frac{x}{4}\right)^2 + \ldots\right)$

$= \frac{1}{2}\left(1 - \frac{1}{8}x + \frac{3}{128}x^2 + \ldots\right)$

$= \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 + \ldots$

6 (b) Hence, find the first three terms of the binomial expansion of $\frac{1}{\sqrt{4-x^3}}$

[2 marks]

substitute $x = -x^3$ in a)

$= \frac{1}{2} - \frac{1}{16}(-x^3) + \frac{3}{256}(-x^3)^2 + \ldots$

$= \frac{1}{2} + \frac{1}{16}x^3 + \frac{3}{256}x^6 + \ldots$

Question 6 continues on the next page

6 (c) Using your answer to part **(b)**, find an approximation for $\int_0^1 \frac{1}{\sqrt{4-x^3}}\,dx$, giving your answer to seven decimal places.

[3 marks]

$$= \int_0^1 \frac{1}{2} + \frac{x^3}{16} + \frac{3x^6}{256}\,dx$$

$$= \left[\frac{1}{2}x + \frac{x^4}{64} + \frac{3x^7}{1792}\right]_0^1$$

$$= \left(\frac{1}{2} + \frac{1}{64} + \frac{3}{1792}\right) - (0 + 0 + 0)$$

$$= 0.5172991 \quad (7dp)$$

6 (d) (i) Edward, a student, decides to use this method to find a more accurate value for the integral by increasing the number of terms of the binomial expansion used.

Explain clearly whether Edward's approximation will be an overestimate, an underestimate, or if it is impossible to tell.

[2 marks]

Each term in expansion will be positive.

If we add another term, answer will increase.

$\therefore$ Edward's approximation is an underestimate

6 **(d) (ii)** Edward goes on to use the expansion from part **(b)** to find an approximation for $\int_{-2}^{0} \frac{1}{\sqrt{4-x^3}}\,dx$

Explain why Edward's approximation is invalid.

[2 marks]

Expansion valid for

$|x| < \sqrt[3]{4}$

$|x| < 1.587401$

$\therefore$ Lower limit $x = -2$ is invalid as $|x| = 2$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$n! = n(n-1)\ldots 1$

$(n-2)! = (n-2)\ldots 1$

7 (a) Using ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ show that ${}^nC_2 = \dfrac{n(n-1)}{2}$ **[2 marks]**

$r = 2,\quad {}^nC_2 = \dfrac{n(n-1)(n-2)\ldots 1}{2! \times (n-2)\ldots 1}$

$= \dfrac{n(n-1)}{2} \dfrac{\times (n-2)\ldots 1}{\times (n-2)\ldots 1}$

which cancels to $\dfrac{n(n-1)}{2}$

7 (b) (i) Show that the equation

$$2 \times {}^nC_4 = 51 \times {}^nC_2$$

simplifies to

$$n^2 - 5n - 300 = 0$$

[3 marks]

${}^nC_2 = \dfrac{n(n-1)}{2}$

$C_4 = \dfrac{n(n-1)(n-2)(n-3)}{4!} \dfrac{\times (n-4)\ldots 1}{(n-4)\ldots 1}$

${}^nC_4 = \dfrac{n(n-1)(n-2)(n-3)}{24}$

$\therefore\ \cancel{2}^{1} \times \dfrac{\cancel{n(n-1)}(n-2)(n-3)}{\cancel{24}_{12}} = \dfrac{51\,\cancel{n(n-1)}}{2}$

$(n-2)(n-3) = \dfrac{12 \times 51}{2}$

$n^2 - 3n - 2n + 6 = 306$

$n^2 - 5n - 300 = 0$

as required

7 (b) (ii) Hence, solve the equation

$$2 \times {}^{n}C_4 = 51 \times {}^{n}C_2$$

[2 marks]

$n^2 - 5n - 300 = 0$

$(n + 15)(n - 20) = 0$

$n = -15$ or $n = 20$

as $n > 0$

$n = 20$