1 -

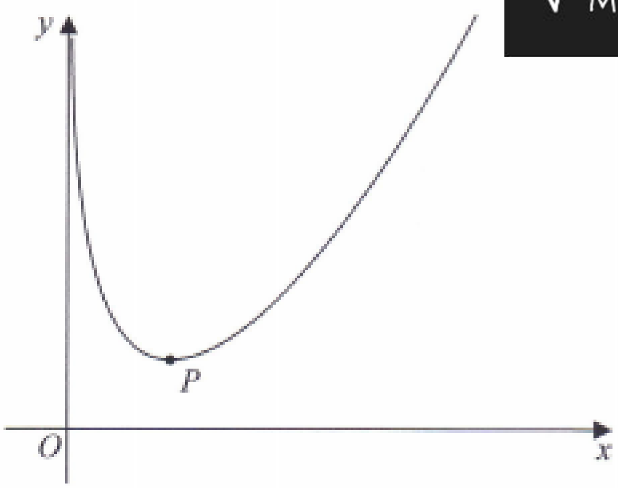


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

a) $y = \dfrac{4x^2 + x}{2\sqrt{x}}$ $\leftarrow u$ $\leftarrow v$ quotient rule

$u = 4x^2 + x$ $\quad v = 2x^{\frac{1}{2}}$

$\dfrac{du}{dx} = 8x + 1$ $\quad \dfrac{dv}{dx} = x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{2x^{\frac{1}{2}}(8x+1) - x^{-\frac{1}{2}}(4x^2+x)}{4x}$$

$$= \frac{16x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 4x^{\frac{3}{2}} - x^{\frac{1}{2}}}{4x}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} \quad 1$$

$$\frac{d}{dx}(-4\ln x) = -\frac{4}{x} \quad 2$$

Adding 1 and 2

$$\frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} - \frac{4}{x}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} - \frac{4}{x} \times \frac{4}{4}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16}{4x}$$

$$= \frac{x^{-\frac{1}{2}}(12x^2 + x - 16x^{\frac{1}{2}})}{4x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

b) at P $\frac{dy}{dx} = 0$

$$0 = 12x^2 + x - 16\sqrt{x}$$

$$0 = x^{\frac{1}{2}}\left(12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16\right)$$

as $x^{\frac{1}{2}} \neq 0$

equals zero when

$$12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16 = 0$$

$$12x^{\frac{3}{2}} + x^{\frac{1}{2}} = 16$$

$$12x^{\frac{3}{2}} = 16 - \sqrt{x}$$

$$x^{\frac{3}{2}} = \frac{16}{12} - \frac{\sqrt{x}}{12}$$

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

as required

c) $x_1 = 2$

(i) $x_2 = \left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^{\frac{2}{3}} = 1.13894$

(ii) $x_2 = 1.138935342$

$x_3 = 1.156928424$

$x_4 = 1.156494678$

$x_5 = 1.156505095$

$x_6 = 1.156504845$

$x_7 = 1.156504851$

rounded to 5dp

1.15650

4. The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx$$

giving your answer to 3 significant figures.

(3)

(b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx$

(1)

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

a) $= \frac{1}{2} \times 0.5 \times \left[0.5774 + 0.8452 + 2(0.7071 + 0.7746 + 0.8165)\right]$

$= 1.50475$

$= 1.50$ (3sf)

b) $\sqrt{9} = 3$ can be taken outside as a constant

$3 \times 1.50 = 4.50$

c) 4.50 is accurate to 2 significant figures

5.

The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

(a) Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$$

(3)

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

a) $f(x) = 2x^3 + x^2 - 1$

$f'(x) = 6x^2 + 2x$

Newton Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \quad x_{n+1} = x_n - \frac{(2x_n^3 + x_n^2 - 1)}{(6x_n^2 + 2x_n)}$$

$$= \frac{x_n(6x_n^2 + 2x_n) - (2x_n^3 + x_n^2 - 1)}{6x_n^2 + 2x_n}$$

$$= \frac{6x_n^3 + 2x_n^2 - 2x_n^3 - x_n^2 + 1}{6x_n^2 - 2x_n}$$

$$= \frac{4x_n^3 + x_n^3 + 1}{6x_n^2 - 2x_n}$$

as required

b) $x_2 = 1 - \dfrac{(2\times 1^3 + 1^2 - 1)}{6\times 1^2 + 2\times 1}$

$x_2 = \dfrac{3}{4}$

$x_3 = \dfrac{3}{4} - \dfrac{(2\times(\frac{3}{4})^3 + (\frac{3}{4})^2 - 1}{6\times(\frac{3}{4})^2 + 2\times\frac{3}{4}}$

$x_3 = \dfrac{2}{3}$

c) The equation of the tangent
$f'(x) = 6x_n^2 + 2x_n$

if $x_1 = 0$, gradient of tangent is zero, so tangent would run parallel to, thus never meet the x-axis.
The Newton Raphson method relies on the tangent intersecting with the x-axis to get the next x value.

6.

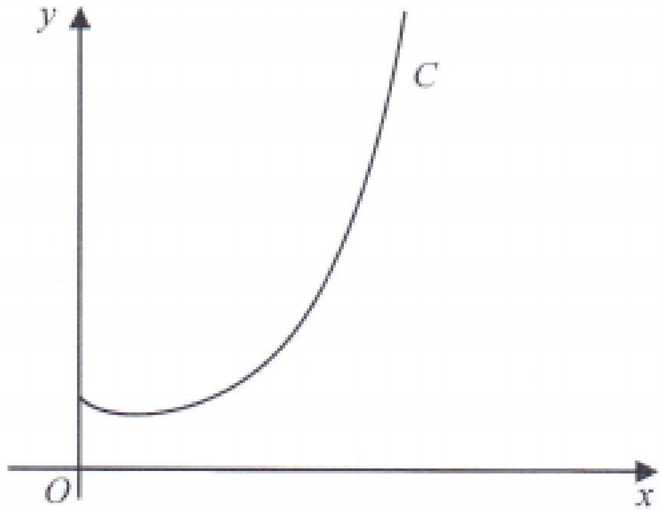


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $\ln y = \ln x^x$

$\ln y = x \ln x$

differentiating both sides

$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = \frac{1 + \ln x}{\frac{1}{y}}$

turning point, $\frac{dy}{dx} = 0$

$$0 = \frac{1 + \ln x}{\frac{1}{y}}$$

$$\therefore \ln x + 1 = 0$$

$$\ln x = -1$$

$$x = e^{-1}$$

$$x = \frac{1}{e} \quad \text{at point C}$$

b) $x_{n+1} = 2x_n^{1-x_n}$

Calculator
$1.5 =$
$2 \times ANS^{1-ANS}$

$x_1 = 1.5$

$x_2 = 1.63299$

$x_3 = 1.46626$

$x_4 = 1.673135$

$x_4 = 1.673$ (3dp)

c) Repeated use of calculator, roots oscillate between 1 and 2 after x_{18}

8. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in $m s^{-1}$.

Time (s)	0	5	10	15	20	25
Speed ($m s^{-1}$)	2	5	10	18	28	42

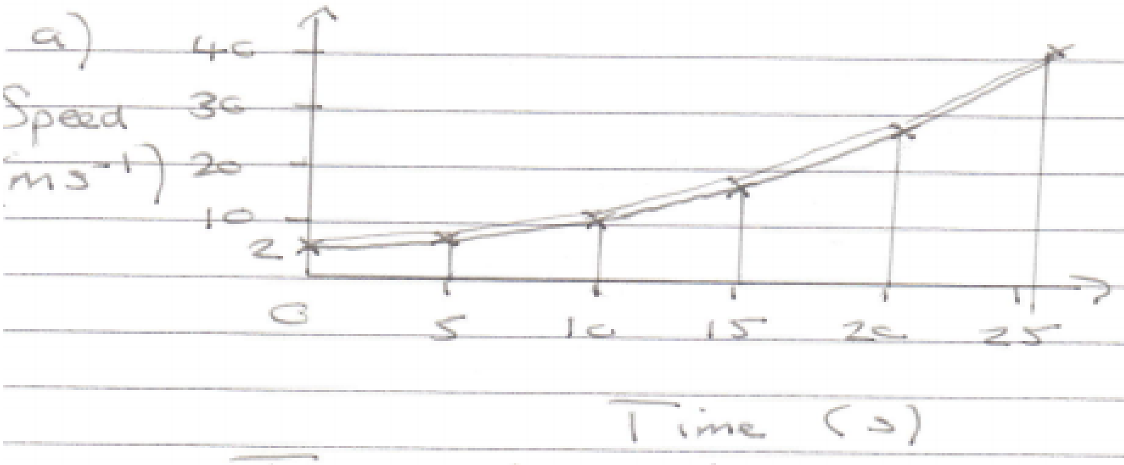
Using all of this information,

(a) estimate the length of runway used by the jet to take off. **(3)**

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off. **(1)**

a)



Trapezium rule to get area under the curve which is the distance

$$\text{Dist} = \frac{1}{2} \times 5\left(2 + 42 + 2(5 + 10 + 18 + 28)\right)$$

$$= 415 \text{ m}$$

b) An overestimate. From graph, the trapezia are above the curve of the graph

9. The curve with equation $y = 2\ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

(a) Show that $3 < \alpha < 4$

(2)

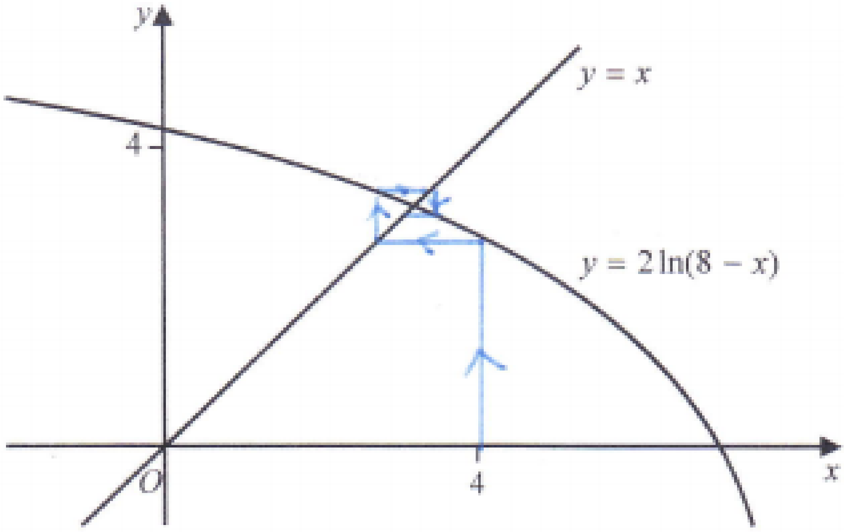


Figure 2

Figure 2 shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$.

A student uses the iteration formula

$$x_{n+1} = 2\ln(8 - x_n), \quad n \in \mathbb{N}$$

in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

(b) determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer.

(2)

a) $y = 2\ln(8-x)$ $\quad y = x$

$\therefore \quad x = 2\ln(8-x)$

$0 = 2\ln(8-x) - x$

$f(3) = 2\ln(8-3) - 3 = 0.218875$

$f(4) = 2\ln(8-4) - 4 = -1.2274$

as there is a change of sign, there is a root in the interval $[3,4]$

$\therefore \quad 3 < \alpha < 4$

b) $x_1 = 4$

Then produced a COBWEB diagram

This iteration can be used as the COBWEB diagram converges to a root