YORKSHIRE MATHS TUTOR

1.

A curve C has equation $y = f(x)$

Given that

- $f'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $f(x)$

find, in simplest form, $f(x)$

(6)

$f'(x) = 6x^2 + ax - 23$

$f(x) = 2x^3 + \frac{1}{2}ax^2 - 23x + c$

at y intercept -12

$-12 = c$

$f(x) = 2x^3 + \frac{1}{2}ax^2 - 23x - 12$

if $(x+4)$ is a factor, $f(-4) = 0$

$2(-4)^3 + \frac{1}{2} \times a(-4)^2 - 23(-4) - 12 = 0$

$-128 + 8a + 92 - 12 = 0$

$8a = 48$

$a = 6$

$f(x) = x^3 + 3x^2 - 23x - 12$

2.

(a) Given that

$$\frac{x^2+8x-3}{x+2} \equiv Ax + B + \frac{C}{x+2} \qquad x \in \mathbb{R} \quad x \neq -2$$

find the values of the constants A, B and C

(3)

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2+8x-3}{x+2}\,dx$$

giving your answer in the form $a + b\ln 2$ where a and b are integers to be found.

(4)

a)

$$\begin{array}{r} x + 6 \\ x+2 \overline{\smash{)} x^2+8x-3} \\ -\underline{x^2+2x} \\ 6x-3 \\ -\underline{6x+12} \\ -15 \end{array}$$

$$\frac{x^2+8x-3}{x+2} = x + 6 \; \frac{-15}{x+2}$$

$A = 1, \; B = 6, \; C = -15$

b) $\displaystyle\int_0^6 x + 6 \; \frac{-15}{x+2} \, dx$

$$= \left[\frac{x^2}{2} + 6x - 15\ln(x+2)\right]_0^6$$

$$= (18 + 36 - 15\ln 8) - (0 + 0 - 15\ln 2)$$

$$= 54 - 15\ln 2^3 + 15\ln 2$$

$$= 54 - 45\ln 2 + 15\ln 2$$

$$= 54 - 30\ln 2$$

3. Show that

$$\int_0^2 2x\sqrt{x+2}\,dx = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

Integrate by parts

$u = 2x \qquad v = \frac{2}{3}(x+2)^{\frac{3}{2}}$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = (x+2)^{\frac{1}{2}}$

$= \left[\frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{4}{3}\int (x+2)^{\frac{3}{2}}\,dx\right]_0^2$

$= \left[\frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{4}{3}\times\frac{2}{5}(x+2)^{\frac{5}{2}}\right]_0^2$

$= \left(\frac{4}{3}\times 2(2+2)^{\frac{3}{2}} - \frac{8}{15}(2+2)^{\frac{5}{2}}\right)$

$- \left(0 - \frac{8}{15}(0+2)^{\frac{5}{2}}\right)$

$= \left(\frac{64}{3} - \frac{256}{15}\right) - \left(0 - \frac{8}{15}\times(\sqrt{2})^5\right)$

$= \frac{64}{15} - \frac{8}{15}\times 4\sqrt{2}$

$(\sqrt{2})^2(\sqrt{2})^2\times\sqrt{2}$
$= 4\sqrt{2}$

$= \frac{64}{15} - \frac{32}{15}\sqrt{2}$

$= \frac{32}{15}\left(2+\sqrt{2}\right)$ as required

4. Given that $k \in \mathbb{Z}^+$

(a) show that $\int_k^{3k} \frac{2}{(3x-k)} \, dx$ is independent of k,

(4)

(b) show that $\int_k^{2k} \frac{2}{(2x-k)^2} \, dx$ is inversely proportional to k.

(3)

a) $\left[\frac{2}{3}\ln(3x-k)\right]_k^{3k}$

$= \left(\frac{2}{3}\ln(9k-k)\right) - \left(\frac{2}{3}\ln(3k-k)\right)$

$= \frac{2}{3}\ln(8k) - \frac{2}{3}\ln 2k$

$= \frac{2}{3}\ln\frac{8k}{2k}$

$= \frac{2}{3}\ln 4$

b) $\int_k^{2k} 2(2x-k)^{-2} \, dx$

$= \left[\frac{2}{-1\times 2}(2x-k)^{-1}\right]_k^{2k}$

$= \left[\frac{-1}{2x-k}\right]_k^{2k}$

$= \left(\frac{-1}{4k-k}\right) - \left(\frac{-1}{2k-k}\right)$

$= -\frac{1}{3k} + \frac{1}{k}$

$= \frac{-1+3}{3k} = \frac{2}{3k}$

$\therefore \propto \frac{1}{k}$

inversely proportional to k

6. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

a) $x = u^2 + 1$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

limits

$x = 5$: $5 = u^2 + 1$, $5 - 1 = u^2$, $4 = u^2$, $u = 2$

$x = 10$: $10 = u^2 + 1$, $10 - 1 = u^2$, $9 = u^2$, $u = 3$

$$\int_5^{10} \frac{3 \quad \overbrace{dx}^{2u\,du}}{(\underbrace{x}_{u^2+1}-1)(3+2\sqrt{\underbrace{x}_{u^2+1}-1})}$$

$$= \int_2^3 \frac{3 \times 2u\,du}{(u^2+1-1)(3+2\sqrt{u^2+1-1})}$$

$$= \int_2^3 \frac{6\cancel{u}\,du}{u^{\cancel{2}}(3+2u)}$$

$$= \int_2^3 \frac{6\,du}{u(3+2u)}$$ as required

b) $$\int_2^3 \frac{6}{u(3+2u)}\,du$$

Partial fraction

$$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u}$$

$$6 = A(3+2u) + Bu$$

$u=0$, $6 = 3A \Rightarrow A = 2$

$u = -\frac{3}{2}$, $6 = -\frac{3}{2}B \Rightarrow B = -4$

$$\int_2^3 \frac{2}{u} + \frac{-4}{3+2u}\,du$$

$$= \Big[2\ln u - 2\ln(3+2u)\Big]_2^3 du$$

$$= (2\ln 3 - 2\ln 9) - (2\ln 2 - 2\ln 7)$$

$$= \ln 3^2 - \ln 9^2 - \ln 2^2 + \ln 7^2$$

$$= \ln 9 - \ln 81 - \ln 4 + \ln 49$$

$$= \ln \frac{9 \times 49}{4 \times 81} = \ln \frac{49}{36}$$

$$= \ln a$$

where $a = \frac{49}{36}$

7. A large spherical balloon is deflating.

Variables $V \quad r \quad t$

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate. c

(a) Using this model, show that

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

a) $\frac{dV}{dt} = -c$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ ①

$V = \frac{4}{3}\pi r^3$ (volume of sphere)

$\frac{dV}{dr} = 4\pi r^2 \quad \therefore \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$

in ①

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times -c = \frac{-c}{4\pi} \times \frac{1}{r^2}$

$\frac{dr}{dt} = -\frac{k}{r^2}$

where k replaces constant $\frac{c}{4\pi}$

b) $t=0 \quad r=40$
$t=5 \quad r=20$

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

$$\int r^2\,dr = \int -k\,dt$$

$$\frac{r^3}{3} = -kt + c$$

$t=0,\ r=40$

$$\frac{64000}{3} = c$$

$$\frac{8000}{3} = -5k + \frac{64000}{3} \qquad t=5,\ r=20$$

$$5k = \frac{56000}{3}$$

$$k = \frac{11200}{3}$$

$$\therefore \quad \frac{r^3}{3} = -\frac{11200}{3}t + \frac{64000}{3}$$

$$r^3 = -11200t + 64000$$

c) valid when $64000 - 11200t \geqslant 0$

$$11200t \leq 64000$$

$$t \leq \frac{40}{7} \text{ seconds}$$

8.

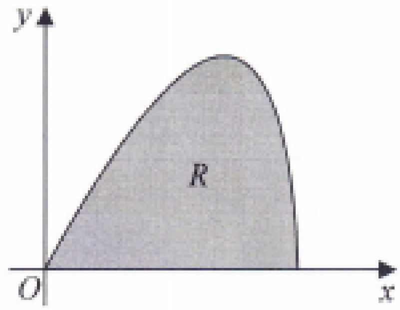


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

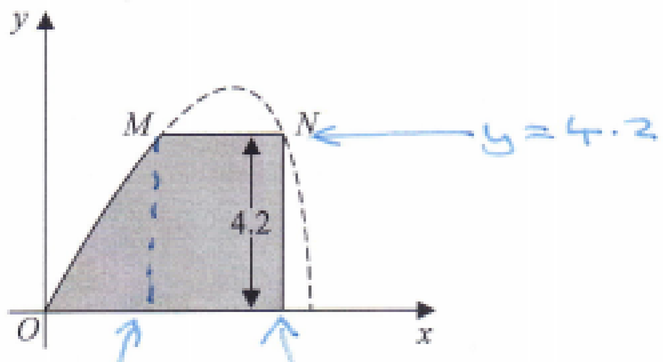
The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(a) (i) Show that the area of R is given by $\displaystyle\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t \,\mathrm{d}t$

(3)

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)



x_1 **Figure 4** $x_2 = 5.293999$

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4. Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

(b) calculate the width of the walkway.

(5)

a) (i) $x = 6\sin t$ $\qquad y = 5\sin 2t$

① $\frac{dx}{dt} = 6\cos t$ $\qquad \frac{dy}{dt} = 10\cos 2t$

$$\text{Area } R = \int y \, dx$$

① gives $dx = 6\cos t \, dt$

$$\text{Area } R = \int_0^{\frac{\pi}{2}} 5\sin 2t \times 6\cos t \, dt$$

$$= \int_0^{\frac{\pi}{2}} 5 \times 2\sin t\cos t \times 6\cos t \, dt$$

$$= \int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t \, dt$$

as required

(ii) $\frac{d}{dt}(\cos^3 t) = -3\cos^2 t\sin t$

$\therefore \frac{d}{dt}(-20\cos^3 t) = 60\sin t\cos^2 t$

$$\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t \, dt = \left[-20\cos^3 t\right]_0^{\frac{\pi}{2}}$$

$$R = (-20 \times 0^3) - (-20 \times 1^3) = 20$$

b) Find x-coordinate where line $y = 4.2$ meets the curve

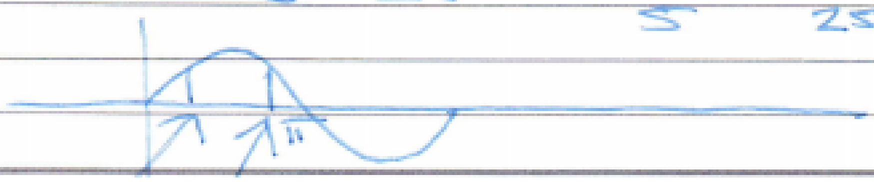
$y = 5\sin 2t$ $\qquad 0 \leq t \leq \frac{\pi}{2}$

$4.2 = 5\sin 2t$

$\sin 2t = \frac{4.2}{5} = \frac{21}{25}$ $\quad 0 \leq 2t \leq \pi$

$2t = 0.99728$

or 2.14309



0.99728 $\qquad$ 2.144309

$\therefore \div$ by 2 $\quad t = 0.4986416$ or $t = 1.0721547$

$x = 6 \sin t$

$x_1 = 6 \sin(0.4986416) = 2.86939799$

or $x_2 = 6 \sin(1.0721547) = 5.26939799$

$MN = x_2 - x_1$

$MN = 5.26939799 - 2.86939799$

$MN = 2.4$ m