2. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \qquad x > 0$$

(a) Find (i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

a)(i) $y = x^2 - 2x - 24x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$

(ii) $\dfrac{d^2y}{dx^2} = 2 + 6x^{-\frac{3}{2}}$

b) $\dfrac{dy}{dx} = 0$ at stationary point

at $x = 4$, $\dfrac{dy}{dx} = 2 \times 4 - 2 - 12 \times \dfrac{1}{4^{\frac{1}{2}}}$

$= 8 - 2 - 6 = 0$

as $\dfrac{dy}{dx} = 0$, stationary point at $x = 4$

c) at $x = 4$, $\dfrac{d^2y}{dx^2} = 2 + 6 \times 4^{-\frac{3}{2}}$

$\dfrac{d^2y}{dx^2} = \dfrac{11}{4}$

as $\dfrac{d^2y}{dx^2} > 0$ this is a minimum point

3.

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$$

(a) Find the values of the constants A, B and C.

(4)

$$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)} \qquad x > 3$$

(b) Prove that f(x) is a decreasing function.

(3)

a) $(x-3)(1-2x) = x - 2x^2 - 3 + 6x$

$= -2x^2 + 7x - 3$

$$\begin{array}{r|l} & 3 \\ -2x^2+7x-3 & -6x^2+11x+1 \\ - & -6x^2+21x-9 \\ \hline & -10x+10 \end{array}$$

$\therefore A = 3$ and

$$\frac{-10x+10}{(x-3)(1-2x)} = \frac{B}{x-3} + \frac{C}{1-2x}$$

$-10x + 10 = B(1-2x) + C(x-3)$

Let $x = 3$, $-30 + 10 = B(1-6)$

$-20 = -5B$

$B = 4$

Let $x = \frac{1}{2}$, $-10 \times \frac{1}{2} + 10 = C(\frac{1}{2} - 3)$

$5 = -2.5C$

$C = -2$

$A = 3$, $B = 4$, $C = -2$

b) $f(x) = 3 + 4(x-3)^{-1} - 2(1-2x)^{-1}$

$f'(x) = -4(x-3)^{-2} + 2 \times -2(1-2x)^{-2}$

$f'(x) = -\dfrac{4}{(x-3)^2} - \dfrac{4}{(1-2x)^2}$

as $x > 3$ always +ve

squared means always +ve

$\therefore -\dfrac{4}{(x-3)^2}$ is always negative

and $-\dfrac{4}{(1-2x)^2}$ is always negative

so $f'(x)$ is always negative for $x > 3$

so $f(x)$ is a decreasing function

5.

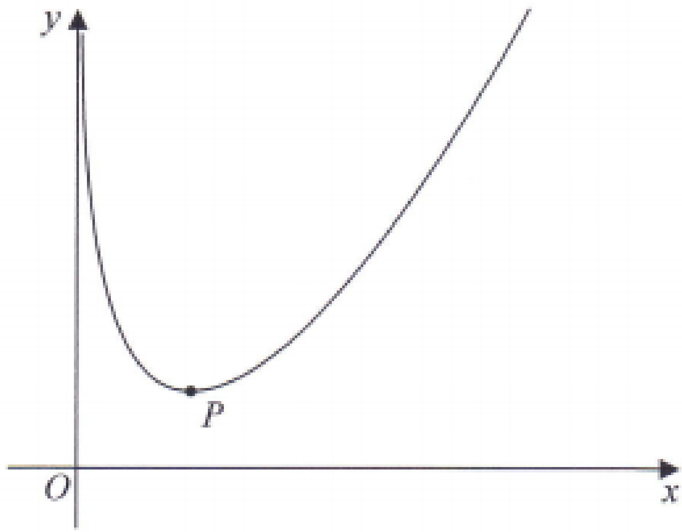

Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

a) $y = \dfrac{4x^2 + x}{2\sqrt{x}}$ $\begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$ quotient rule

$u = 4x^2 + x \qquad v = 2x^{\frac{1}{2}}$

$\dfrac{du}{dx} = 8x + 1 \qquad \dfrac{dv}{dx} = x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{2x^{\frac{1}{2}}(8x+1) - x^{-\frac{1}{2}}(4x^2+x)}{4x}$$

$$= \frac{16x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 4x^{\frac{3}{2}} - x^{\frac{1}{2}}}{4x}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} \quad ①$$

$$\frac{d}{dx}(-4\ln x) = -\frac{4}{x} \quad ②$$

Adding ① and ②

$$\frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} - \frac{4}{x}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}}}{4x} - \frac{4}{x} \times \frac{4}{4}$$

$$= \frac{12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16}{4x}$$

$$= \frac{x^{-\frac{1}{2}}(12x^2 + x - 16x^{\frac{1}{2}})}{4x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

as required

Turn over

6.

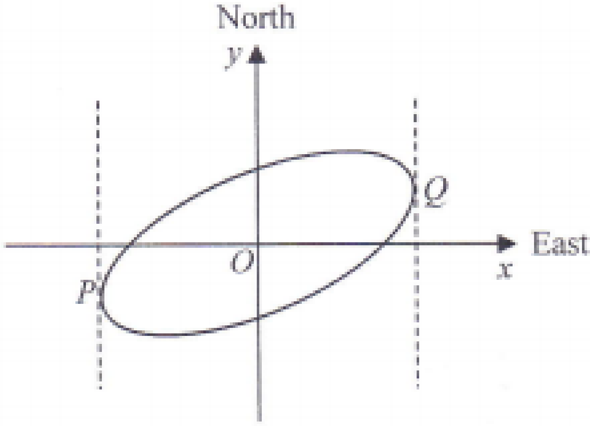

Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

(a) Show that $\frac{dy}{dx} = \frac{y-x}{3y-x}$ **(4)**

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P. **(5)**

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O. (You **do not** need to carry out this calculation). **(1)**

a) $x^2 - 2xy + 3y^2 = 50$

Implicit differentiation

$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$

$2x - 2y = 2x\frac{dy}{dx} - 6y\frac{dy}{dx}$

$2x - 2y = \frac{dy}{dx}(2x - 6y)$

$\frac{dy}{dx} = \frac{2x-2y}{2x-6y} = \frac{\cancel{2}(x-y)}{\cancel{2}(x-3y)}$

$$\frac{dy}{dx} = \frac{x-y}{x-3y} \begin{matrix}(x \text{ by } -1)\\(x \text{ by } -1)\end{matrix}$$

$$\frac{dy}{dx} = \frac{y-x}{3y-x} \quad \text{as required}$$

b) at point P $\frac{dx}{dy} = 0$ (vertical tangent)

$$\therefore \frac{dx}{dy} = \frac{3y-x}{y-x} = 0$$

$$\therefore x = 3y$$

$$x^2 - 2xy + 3y^2 = 50$$
$$(3y)^2 - 2(3y)y + 3y^2 = 50$$

$$9y^2 - 6y^2 + 3y^2 = 50$$
$$6y^2 = 50$$
$$y^2 = \frac{50}{6}$$

$$y = \sqrt{\frac{50}{6}} = \pm\frac{5\sqrt{3}}{3}$$

$$x = 3y = \pm 5\sqrt{3}$$

Coordinates of P $\left(-5\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)$

c) find where $\frac{dy}{dx} = 0$

$\therefore$ set $y = x$

solve simultaneous equations

$$x^2 - 2xy + 3y^2 = 5$$

and $y = x$

8.

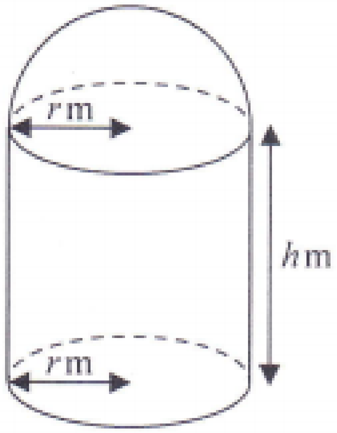

Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is $6\,\text{m}^3$. V h

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)

a) Area of base $= \pi r^2$

Curved area of cylinder (rectangle)

$= 2 \times \pi \times r \times h = 2\pi r h$

Curved hemisphere $= 2\pi r^2$

total $= 3\pi r^2 + 2\pi r h$ (1)

need to eliminate h

V of cylinder $= \pi r^2 h$

V of hemisphere $= \frac{1}{2} \times \frac{4}{3} \pi r^3$

$$\therefore \; 6 = \pi r^2 h + \frac{2}{3} \pi r^3$$

$$6 - \frac{2}{3} \pi r^3 = \pi r^2 h$$

$$\frac{6 - \frac{2}{3} \pi r^3}{\pi r^2} = h$$

sub h in (1)

Total area $= 3\pi r^2 + \dfrac{2\cancel{\pi}\cancel{r} \times \left(6 - \frac{2}{3}\pi r^3\right)}{\cancel{\pi} r^{\cancel{2}}}$

$$= 3\pi r^2 + \frac{12}{r} - \frac{4}{3}\pi r^2$$

$$= \frac{12}{r} + \frac{5}{3}\pi r^2 \quad \text{as required}$$

b) $A = 12 r^{-1} + \frac{5}{3}\pi r^2$

$$\frac{dA}{dr} = \frac{-12}{r^2} + \frac{10}{3}\pi r$$

$$\frac{dA}{dr} = 0 \Rightarrow 0 = \frac{-12}{r^2} - \frac{10}{3}\pi r$$

$$\frac{12}{r^2} = \frac{10}{3}\pi r$$

$$\frac{12 \times 3}{10\pi} = r^3$$

$$r = \sqrt[3]{\frac{12 \times 3}{10\pi}} = 1.046$$

$$= 1.05 \quad (3 \text{sf})$$

c) $A = \dfrac{12}{1.046\ldots} + \dfrac{5}{3} \times \pi \times 1.046\ldots^2$

$$= 17.20105016$$

$$= 17 \, m^2 \quad (\text{nearest integer})$$