2.

Given that θ is measured in radians, prove, from first principles, that

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for $\cos(A \pm B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

let $f(\theta) = \cos\theta$

$$f'(\theta) = \lim_{h\to 0} \frac{f(\theta+h) - f(\theta)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(\theta+h) - \cos\theta}{h}$$

$$= \lim_{h\to 0} \frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$$

$$= \lim_{h\to 0} \left(\left(\frac{\cos h - 1}{h}\right)\cos\theta - \left(\frac{\sin h}{h}\right)\sin\theta\right)$$

$\uparrow 0$ $\uparrow 1$

since $\frac{\cos h - 1}{h} \to 0$

and $\frac{\sin h}{h} \to 1$

$$\lim_{h\to 0}\left(0 \times \cos\theta - \sin\theta\right)$$

$$= -\sin\theta$$

$$\therefore \frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

3. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \qquad x > 0$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 4$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

a)(i) $y = x^2 - 2x - 24x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$

(ii) $\frac{d^2y}{dx^2} = 2 + 6x^{-\frac{3}{2}}$

b) $\frac{dy}{dx} = 0$ at stationary point

at $x=4$, $\frac{dy}{dx} = 2\times4 - 2 - 12\times\frac{1}{4^{\frac{1}{2}}}$

$= 8 - 2 - 6 = 0$

as $\frac{dy}{dx} = 0$, stationary point at $x = 4$

c) at $x = 4$, $\frac{d^2y}{dx^2} = 2 + 6\times4^{-\frac{3}{2}}$

$\frac{d^2y}{dx^2} = \frac{11}{4}$

as $\frac{d^2y}{dx^2} > 0$ this is a minimum point

4. Given that

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1 + \sin 2\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where A is a rational constant to be found.

(5)

$u = 3\sin\theta$

$v = 2\sin\theta + 2\cos\theta$

Quotient rule

$u = 3\sin\theta \qquad v = 2\sin\theta + 2\cos\theta$

$\frac{du}{d\theta} = 3\cos\theta \qquad \frac{dv}{d\theta} = 2\cos\theta - 2\sin\theta$

$\frac{dy}{d\theta} = \frac{v\frac{du}{d\theta} - u\frac{dv}{dv}}{v^2}$

$= \frac{3\cos\theta(2\sin\theta + 2\cos\theta) - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$

$= \frac{\cancel{6\sin\theta\cos\theta} + 6\cos^2\theta - \cancel{6\sin\theta\cos\theta} + 6\sin^2\theta}{4\sin^2\theta + 8\sin\theta\cos\theta + 4\cos^2\theta}$

$= \frac{6\cos^2\theta + 6\sin^2\theta}{4\sin^2\theta + 4\times(2\sin\theta\cos\theta) + 4\cos^2\theta}$

$$= \frac{6(\cos^2\theta + \sin^2\theta)}{4(\sin^2\theta + \cos^2\theta) + 2\sin 2\theta}$$

$$= \frac{6}{4 + 4\sin 2\theta}$$

$$= \frac{6}{4(1 + \sin 2\theta)}$$

$$= \frac{3}{2} \times \frac{1}{1 + \sin 2\theta}$$

$$= \frac{\frac{3}{2}}{1 + \sin 2\theta}$$

where $A = \frac{3}{2}$

5.

$$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$ where A and n are constants to be found. **(4)**

(b) Hence deduce the range of values for x for which $\dfrac{dy}{dx} < 0$ **(1)**

Quotient rule

$u = 5x^2 + 10x \qquad v = (x+1)^2$

$\dfrac{du}{dx} = 10x + 10 \qquad \dfrac{dv}{dx} = 2(x+1)$

$$\frac{dy}{dx} = \frac{(10x+10)(x+1)^2 - 2(x+1)(5x^2+10x)}{(x+1)^4}$$

$$= \frac{10(x+1)^3 - 2 \times 5x(x+2)(x+1)}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)}\left[10(x+1)^2 - 10x(x+2)\right]}{(x+1)^{\cancel{4}\,3}}$$

$$= \frac{10(x^2+2x+1) - 10x^2 - 20x}{(x+1)^3}$$

$$= \frac{10x^2 + 20x + 10 - 10x^2 - 20x}{(x+1)^3}$$

$$\frac{dy}{dx} = \frac{10}{(x+1)^3} \qquad \text{where } A = 10, \; n = 3$$

b) $\dfrac{dy}{dx} < 0 \Rightarrow \dfrac{10}{(x+1)^3} < 0$

$\therefore (x+1)^3 < -1$

$\Rightarrow x < -1$

6.

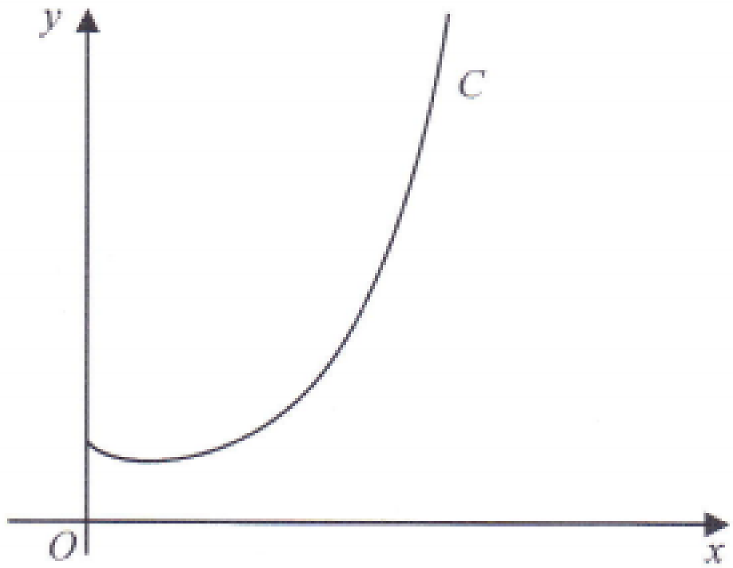

Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x,\ x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $\ln y = \ln x^x$

$\ln y = x \ln x$

differentiating both sides

$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = \frac{1 + \ln x}{\frac{1}{y}}$

turning point, $\frac{dy}{dx} = 0$

$$0 = \frac{1 + \ln x}{\frac{1}{y}}$$

$\therefore \ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$

$\underline{x = \frac{1}{e}}$ at point C

b) $x_{n+1} = 2x_n^{1-x_n}$

Calculator
$1.5 =$
$2 \times ANS^{1-ANS}$

$x_1 = 1.5$

$x_2 = 1.63299$

$x_3 = 1.46626$

$x_4 = 1.673135$

$x_4 = 1.673$ (3dp)

c) Repeated use of calculator, roots oscillate between 1 and 2 after x_{18}

7.

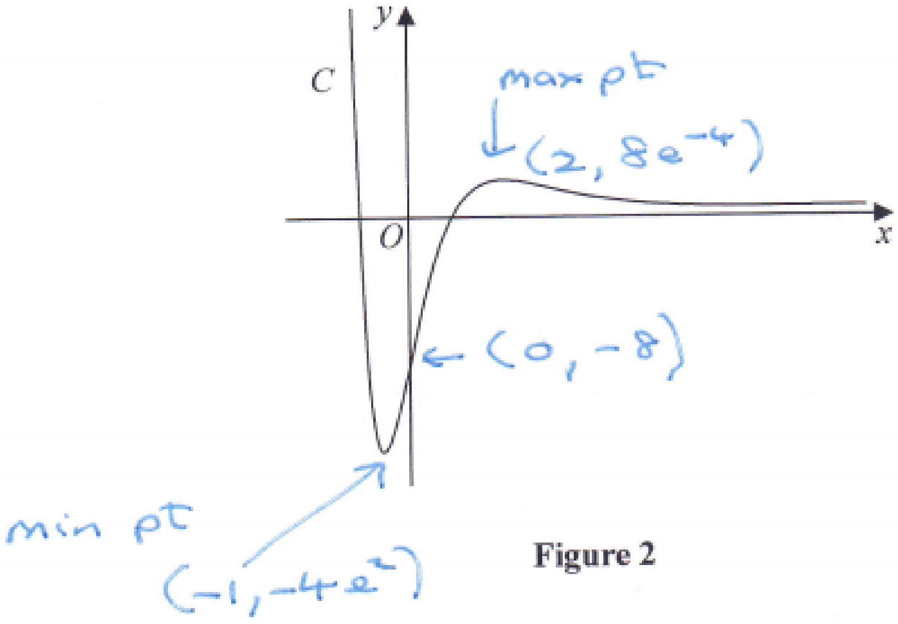

Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)e^{-2x} \qquad x \in \mathbb{R}$$

(a) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)e^{-2x}$ **(3)**

(b) Hence find, in simplest form, the exact coordinates of the stationary points of C. **(3)**

The function g and the function h are defined by

$$\mathrm{g}(x) = 2\mathrm{f}(x) \qquad x \in \mathbb{R}$$

$$\mathrm{h}(x) = 2\mathrm{f}(x) - 3 \qquad x \geqslant 0$$

(c) Find (i) the range of g

(ii) the range of h **(3)**

a) $f(x) = 4(x^2 - 2)\,e^{-2x}$

u $\qquad$ v

$u = 4x^2 - 8 \qquad v = e^{-2x}$

$u' = 8x \qquad v' = -2e^{-2x}$

$f'(x) = 8xe^{-2x} - 8x^2e^{-2x} + 16e^{-2x}$

$= 8e^{-2x}(x - x^2 + 2)$

as required

b) stationary point when
$f'(x) = 0$
$0 = 8e^{-2x}(-x^2+x+2)$
cannot
be
zero

$\therefore \; -x^2+x+2 = 0$

$(-x+2)(x+1) = 0$
$x = 2$ or $x = -1$

$\therefore \; f(2) = 4(2^2-2)e^{-4} = 4 \times 2e^{-4}$
$= 8e^{-4}$
$f(-1) = 4((-1)^2-2)e^{2} = 4 \times -1 \times e^2$
$= -4e^2$
Stationary points are

$(2, 8e^{-4})$ and $(-1, -4e^2)$

c) (i) $g(x) = 2f(x) = 8(x^2-2)e^{-2x}$
stretch in y direction x2

min pt was $(-1, -4e^2)$
$\therefore$ minpt $g(x) = (-1, -8e^2)$
range is $g(x) \geq -8e^2$

(ii) $h(x) = 2f(x) - 3$ ← then translate 3 units down
y coords x2

this is for $x \geq 0$

2 transformations performed

10. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1) **(1)**

a) at $t = 0$

$$N = \frac{900}{3+7e^{-0.25\times 0}} = 90$$

b) $N = 900(3+7e^{-0.25t})^{-1}$

$$\frac{dN}{dt} = 900 \times -1\,(3+7e^{-0.25t})^{-2} \times (7 \times -0.25\, e^{-0.25t})$$

$$\frac{dN}{dt} = \frac{(900\times -1\times 7\times -0.25)\,e^{-0.25t}}{(3+7e^{-0.25t})^2}$$

but $N = \frac{900}{3+7e^{-0.25t}}$

$\Rightarrow N(3+7e^{-0.25t}) = 900$

$3N + 7Ne^{-0.25t} = 900$

$$e^{-0.25t} = \frac{900-3N}{7N}$$

$$\therefore \frac{dN}{dt} = \left(\frac{900}{(3+7e^{-0.25t})}\right) \times \frac{\frac{7}{4}e^{-0.25t}}{3+7e^{-0.25t}}$$

$$\uparrow$$

$$N$$

$$\frac{dN}{dt} = N \times \frac{\frac{7}{4}\left(\frac{900-3N}{7N}\right)}{3+7\left(\frac{900-3N}{7N}\right)}$$

$$= N \times \frac{\frac{225}{N} - \frac{3}{4}}{3 + \frac{900-3N}{N}}$$

$$= N \times \left(\frac{\frac{225\times4 - 3N}{4N}}{\frac{3N+900-3N}{N}}\right) = N \times \left(\frac{\frac{900-3N}{4N}}{\frac{900}{N}}\right)$$

$$= N \times \left(\frac{900-3N}{4\cancel{N}}\right) \times \frac{\cancel{N}}{900}$$

$$= N \times \frac{900-3N}{3600} \quad \text{(divide top \& bottom by 3)}$$

$$= N \times \left(\frac{300-N}{1200}\right) \quad \text{as required}$$

c) at max $\frac{d^2N}{dt^2} = 0$

$$\frac{dN}{dt} = \frac{300N}{1200} - \frac{N^2}{1200}$$

11. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{dy}{dx}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(3)

a) $x = 4\sin 2y$

$\frac{dx}{dy} = 8\cos 2y$

$\frac{dy}{dx} = \frac{1}{8\cos 2y}$ at $y = 0$

$\frac{dy}{dx} = \frac{1}{8 \times \cos 0} = \frac{1}{8}$

b) (i) $\sin 2y = 2y$
for small angle approximation

$\therefore x = 4 \times 2y$

$y = \frac{x}{8}$

(ii) Both have a gradient of $\frac{1}{8}$
So line is a tangent to curve at $(0,0)$

14c) $\frac{dy}{dx} = \frac{1}{8\cos 2y}$ ① $\qquad x = 4\sin 2y$ ②

Identity $\cos^2 y + \sin^2 y = 1$

$\cos^2 2y + \sin^2 2y = 1$

① gives $\cos 2y = \frac{1}{8\frac{dy}{dx}}$

② gives $\sin 2y = \frac{x}{4}$

$$\cos^2 2y + \sin^2 2y = 1$$

$$\left(\frac{1}{8\frac{dy}{dx}}\right)^2 + \left(\frac{x}{4}\right)^2 = 1$$

$$\frac{1}{64\left(\frac{dy}{dx}\right)^2} + \frac{x^2}{16} = 1$$

$$\frac{1}{64\left(\frac{dy}{dx}\right)^2} = 1 - \frac{x^2}{16}$$

$$\frac{1}{64\left(\frac{dy}{dx}\right)^2} = \frac{16 - x^2}{16}$$

$$\frac{\overset{1}{\cancel{16}}}{\underset{4}{\cancel{64}}(16 - x^2)} = \left(\frac{dy}{dx}\right)^2$$

$$\frac{1}{2\sqrt{16 - x^2}} = \frac{dy}{dx}$$

where $a = 2$

$b = 16$

12. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

a) $x^2 \tan y = 9$ ①

implicitly differentiating

$$x^2 \times \sec^2 y \frac{dy}{dx} + \tan y \times 2x = 0$$

$$x^2 \sec^2 y \frac{dy}{dx} = -2x \tan y$$

$$\frac{dy}{dx} = \frac{-2x \tan y}{x^2 \sec^2 y}$$

rearranging ① gives $\tan y = \frac{9}{x^2}$ ③

Identity $\sec^2 y = 1 + \tan^2 y$ ④

$$= 1 + \frac{81}{x^4}$$

Substituting ③ and ④ in $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2\cancel{x} \times \frac{9}{x^2}}{x^2\left(1 + \frac{81}{x^4}\right)}$$

$$\frac{dy}{dx} = \frac{-18}{x^3\left(\frac{x^4+81}{x^4}\right)}$$

$$\frac{dy}{dx} = \frac{-18}{x^4+81} = \frac{-18x}{x^4+81}$$

as required

b) $u = -18x$ $\quad v = x^4 + 81$

$u' = -18$ $\quad v' = 4x^3$

$$\frac{d^2y}{dx^2} = \frac{-18(x^4+81) + 72x^4}{(x^4+81)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-18x^4 - 1458 + 72x^4}{(x^4+81)^2}$$

$$= \frac{54x^4 - 1458}{(x^4+81)^2}$$

$$= \frac{54(x^4 - 27)}{(x^4+81)^2}$$

We have a point of inflection if $\frac{d^2y}{dx^2} = 0$

$\therefore$ $x^4 - 27 = 0$

$x^4 = 27$

$x = \sqrt[4]{27}$ as required