YORKSHIRE MATHS TUTOR

4. The depth of water, D metres, in a harbour on a particular day is modelled by the formula

$$D = 5 + 2\sin(30t)^\circ \qquad 0 \leqslant t < 24$$

where t is the number of hours after midnight.

A boat enters the harbour at 6:30 am and it takes 2 hours to load its cargo.

(4)

a) $D = 5 + 2\sin(30t)$

enters harbour at 6.30 am
$t = 6.5$ hours

$D = 5 + 2\sin(30 \times 6.5)$
$= 4.48236$
$= 4.48$ m (3sf)

b) 2 hours to load cargo

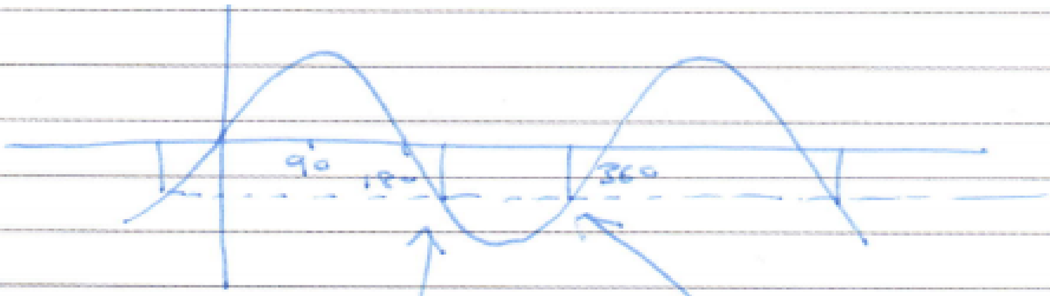
at 8.30 am $t = 8.5$
$D = 5 + 2\sin(30 \times 8.5)$
$= 3.068$ m (too low)

need $D = 3.8$

$3.8 = 5 + 2\sin(30t)$
$\frac{3.8 - 5}{2} = \sin(30t)$

$-\frac{3}{5} = \sin(30t)$

$\sin^{-1}\left(-\frac{3}{5}\right) = -36.86989°$



$360 - 36.869$

$= 323.130$

$180 + 36.869$

$= 216.869$

$30t = 216.869$ $\quad$ $30t = 323.130$

$t = 7.2289$ hrs $\quad$ $t = 10.771$ hrs

before 8.30am

10.771 hrs

= 10 hrs 46 mins

Boat can leave at 10.46 am

5. (a) Solve, for $-180° \leqslant \theta \leqslant 180°$, the equation

$$5\sin 2\theta = 9\tan\theta$$

giving your answers, where necessary, to one decimal place.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

(b) Deduce the smallest positive solution to the equation

$$5\sin(2x - 50°) = 9\tan(x - 25°)$$

(2)

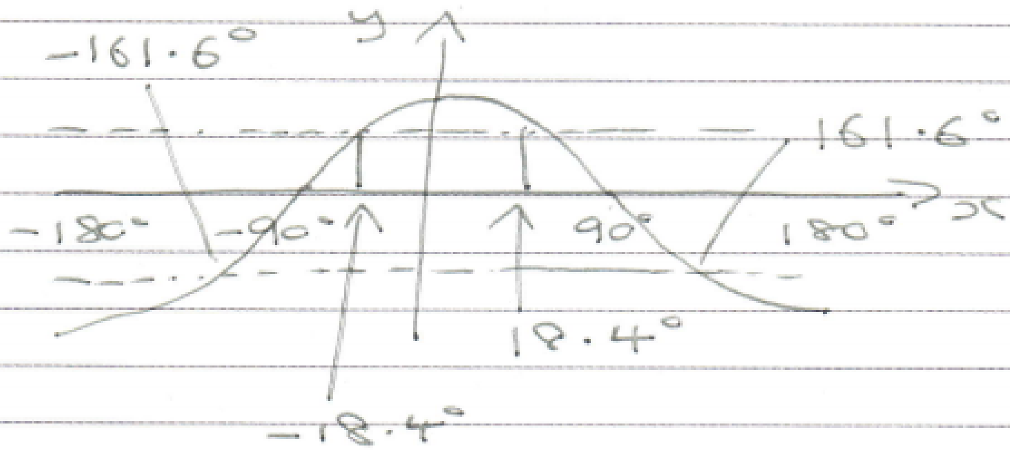
$$5\sin 2\theta = 9\tan\theta$$

$$5 \times 2\cancel{\sin\theta}\cos\theta = 9\,\frac{\cancel{\sin\theta}}{\cos\theta}$$

$$\cos^2\theta = \frac{9}{10}$$

$$\cos\theta = +\sqrt{0.9} \qquad \cos\theta = -\sqrt{0.9}$$

$$\theta = 18.434° \qquad \theta = 161.565$$



$$\theta = -161.6°, -18.4°, 18.4°, 161.6°$$

b) $5\sin(2x - 50°) = 9\tan(x - 25°)$

$5\sin 2(x - 25°) = 9\tan(x - 25°)$

$\therefore\ \theta = x - 25$

$\therefore \quad -161.6 = x - 25 \quad \Rightarrow x = -136.6^\circ$

$-18.4 = x - 25 \quad \Rightarrow x = 6.6^\circ$

$18.4 = x - 25 \quad \Rightarrow x = 43.4^\circ$

$161.6 = x - 25 \quad \Rightarrow x = 186.6$

Smallest solution is $\underline{\underline{x = 6.6^\circ}}$

7.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta \qquad \theta \neq (180n)^\circ \qquad n \in \mathbb{Z}$$

(3)

(b) Hence, or otherwise, solve for $0 < x < 180^\circ$

$$\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50^\circ)$$

(5)

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\therefore \frac{\cos\theta \times \cos\theta}{\sin\theta} = \cos\theta\cot\theta$

$\operatorname{cosec}\theta - \sin\theta$

$= \frac{1}{\sin\theta} - \frac{\sin\theta}{1}$

$= \frac{1 - \sin^2\theta}{\sin\theta}$

$= \frac{\cos^2\theta}{\sin\theta} = \cos\theta \times \frac{\cos\theta}{\sin\theta}$

$= \cos\theta\cot\theta$ (as required)

Identity
$\sin^2\theta + \cos^2\theta = 1$
$1 - \sin^2\theta = \cos^2\theta$

b) $\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50)$

$\therefore \cancel{\cos x}\cot x = \cancel{\cos x}\cot(3x - 50)$

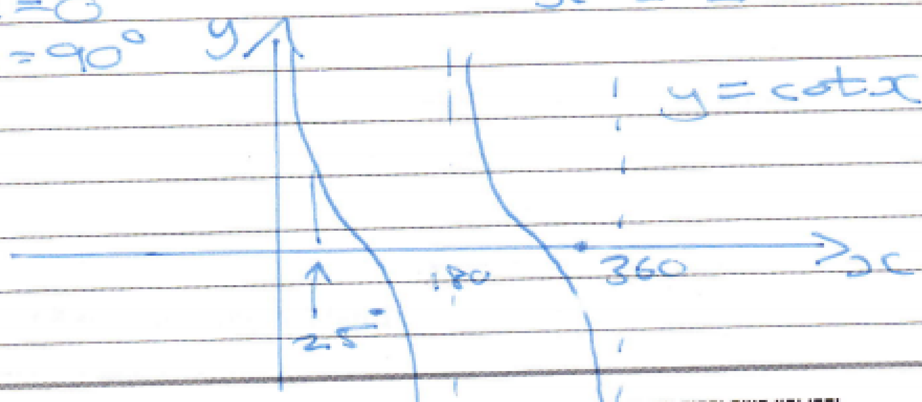
$\therefore x = 3x - 50$

$2x = 50$

$x = 25 \qquad 0 < x < 180^\circ$

$\cos x = 0$

$\Rightarrow x = 90^\circ$



$\cot x = \cot(3x - 50)$

next solution has a period of $180°$

$x + 180 = 3x - 50$

$230 = 2x$

$x = 115°$ in $0 < x < 180$

try $x + 360 = 3x - 50$

$410 = 2x$

$x = 205°$ outside $0 < x < 180$

also when we divided both sides initially by $\cos x$,

$\therefore$ another solution when $\cos x = 0$

$\therefore x = 90°$

Solutions are $x = 25°, 90°, 115°$

8.

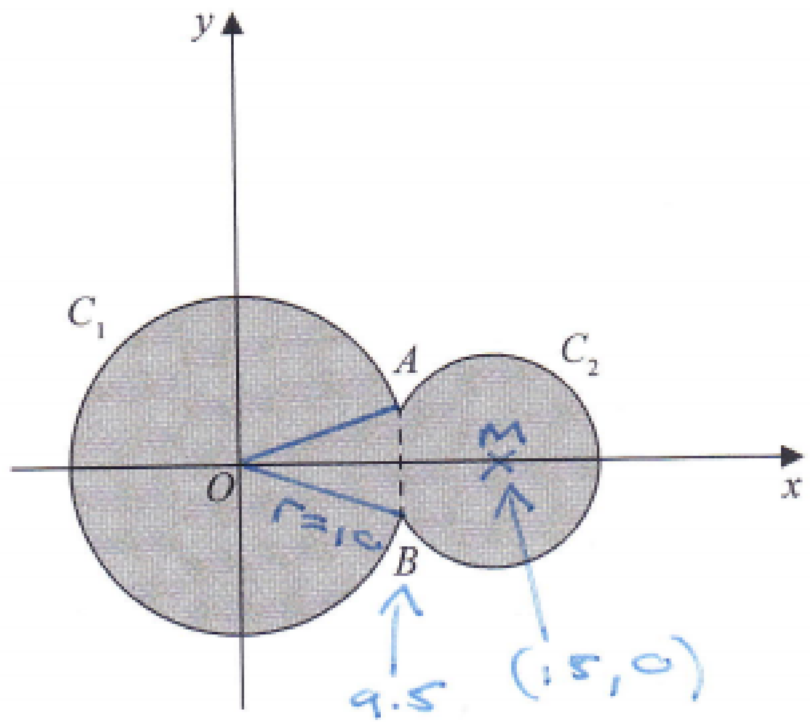


Figure 3

① Circle C_1 has equation $x^2 + y^2 = 100$ $r = 10$, $r = \sqrt{40}$

② Circle C_2 has equation $(x-15)^2 + y^2 = 40$ centre $(15, 0)$

The circles meet at points A and B as shown in Figure 3.

(a) Show that angle $AOB = 0.635$ radians to 3 significant figures, where O is the origin. **(4)**

The region shown shaded in Figure 3 is bounded by C_1 and C_2

(b) Find the perimeter of the shaded region, giving your answer to one decimal place. **(4)**

a) ① and ② gives

$$x^2 + 40 - (x-15)^2 = 100$$

$$x^2 + 40 - (x^2 - 30x + 225) = 100$$

$$\cancel{x^2} + 40 - \cancel{x^2} + 30x - 225 = 100$$

$$30x = 100 + 225 - 40$$

$$30x = 285$$

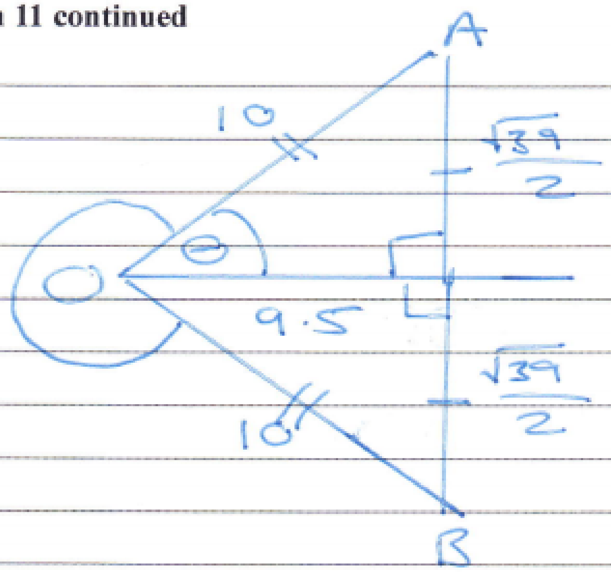
$$x = 9.5$$

in ① $(9.5)^2 + y^2 = 100$

$$y^2 = 100 - 9.5^2$$

$$y = \pm\frac{\sqrt{39}}{2}$$

Question 11 continued



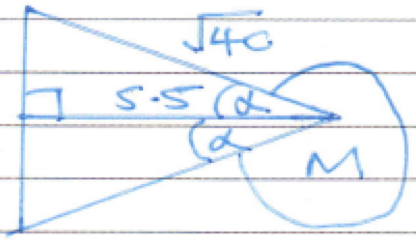
$\cos\theta = \frac{9.5}{10}$

$\theta = 0.31756^r$

$\angle AOB = 2\theta$
$= 0.635^r$
(3sf)

Reflex angle at O $= 2\pi - 0.635120\ldots$
$= 5.64806444\,9^r$

Arc length $C_1 = 10 \times 5.64806444\,9$
$= 56.4806444\,9$ (3)



$2\alpha = 2\cos^{-1}\left(\frac{5.5}{\sqrt{40}}\right)$

$= 1.032702637^r$

reflex angle $= 2\pi - 1.032702637$
$= 5.25048267\,1^r$

C_2 arc length $= \sqrt{40} \times 5.25048267^r$
$= 33.20696811$ (4)

Perimeter $=$ (3) $+$ (4)
$= 89.6876126$
$= 89.7$ units (1dp)

9. (i) Solve, for $0 \leqslant x < \frac{\pi}{2}$, the equation

$$4\sin x = \sec x$$

(4)

(ii) Solve, for $0 \leqslant \theta < 360°$, the equation

$$5\sin\theta - 5\cos\theta = 2$$

giving your answers to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

(i) $4\sin x = \frac{1}{\cos x}$

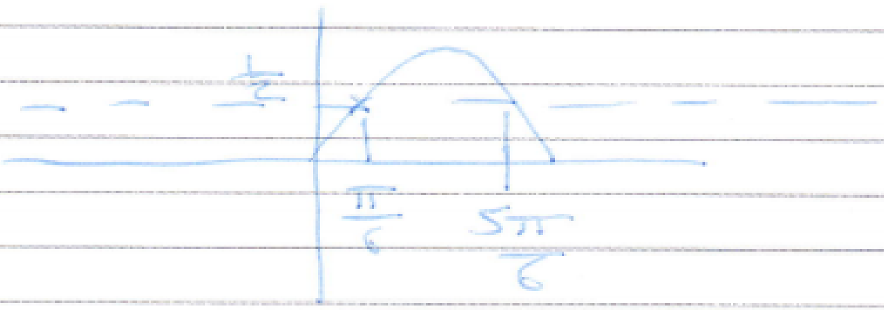
$4\sin x\cos x = 1$

$2\times(2\sin x\cos x) = 1$

$\sin 2x = \frac{1}{2}$

$0 \leq x < \frac{\pi}{2}$

$0 \leq 2x < \pi$



$2x = \frac{\pi}{6}$ $\quad$ $2x = \frac{5\pi}{6}$

$x = \frac{\pi}{12}$ $\quad$ $x = \frac{5\pi}{12}$

(ii) $5\sin\theta - 5\cos\theta = 2$

$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = 5$ $\quad$ $R\sin\alpha = 5$

$\tan\alpha = \frac{5}{5} = 1 \Rightarrow \alpha = 45°$

$R = \sqrt{5^2 + 5^2} = \sqrt{50}$

$\therefore \quad \sqrt{50}\ \sin(\theta - 45) = 2$

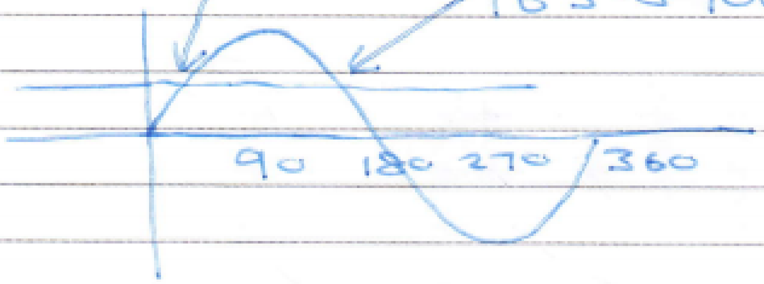
$\sin(\theta - 45) = \dfrac{2}{\sqrt{50}}$

$0 \leq \theta < 360^\circ$

16.42994

$\sin^{-1}\left(\dfrac{2}{\sqrt{50}}\right)$

$163.5700 \qquad = 16.42994$



$\theta - 45 = 16.42994 \Rightarrow \theta = 61.4^\circ$

$\theta - 45 = 163.5700 \Rightarrow \theta = 208.6^\circ$

10.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

(4)

(b) Hence solve, for $-90° \leqslant x \leqslant 180°$, the equation

$$1 - \cos 3x = \sin^2 x$$

(4)

a) $\cos(3A) = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= \underbrace{(2\cos^2 A - 1)}_{\cos 2A}\cos A - \underbrace{(2\sin A\cos A)}_{\sin 2A}\sin A$

$= 2\cos^3 A - \cos A - 2\cos A\sin^2 A$

$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

(as Required)

b) $1 - \cos 3x = 1 - 4\cos^3 x + 3\cos x$

$\therefore \quad 1 - 4\cos^3 x + 3\cos x = \sin^2 x$

$\cancel{1} - 4\cos^3 x + 3\cos x = \cancel{1} - \cos^2 x$

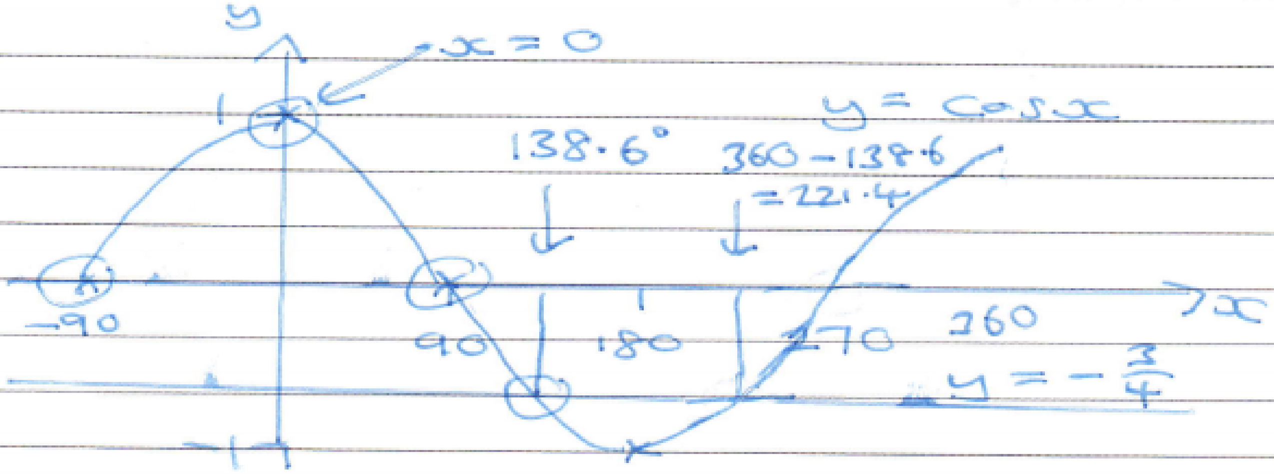
$0 = 4\cos^3 x - \cos^2 x - 3\cos x$

$0 = \cos x(4\cos^2 x - \cos x - 3)$

$0 = \cos x(4\cos x + 3)(\cos x - 1)$

Either $\cos x = 0$, $\cos x = -\frac{3}{4}$, $\cos x = 1$

Question 10 continued



solutions in range are circled

$-90 \leq x \leq 180$

$x = -90, 0, 90, 138.6$