3. In the binomial expansion of

$$(a + 2x)^7 \qquad \text{where } a \text{ is a constant}$$

the coefficient of x^4 is 15 120

Find the value of a.

(3)

$${}^7C_0\, a^7 + {}^7C_1\, a^6 \times 2x + {}^7C_2\, a^5 (2x)^2$$

$$+ {}^7C_3\, a^4 (2x)^3 + {}^7C_4\, a^3 (2x)^4$$

$$+ \ldots$$

$${}^7C_4 \times a^3 \times 2^4 = 15120$$

$$35 \times 16 \times a^3 = 15120$$

$$a^3 = \frac{15120}{35 \times 16}$$

$$a = \sqrt[3]{\frac{15120}{35 \times 16}}$$

$$a = 3$$

4. A car has six forward gears.

The fastest speed of the car

- in 1st gear is 28 km h^{-1}
- in 6th gear is 115 km h^{-1}

Given that the fastest speed of the car in successive gears is modelled by an **arithmetic sequence**,

(a) find the fastest speed of the car in 3rd gear. **(3)**

Given that the fastest speed of the car in successive gears is modelled by a **geometric sequence**,

(b) find the fastest speed of the car in 5th gear. **(3)**

1st	2nd	3rd	4th	5th	6th
28		$a+2d$			115

$a = 28$

$a + 5d = 115$

$\therefore \quad 28 + 5d = 115$

$5d = 115 - 28$

$d = \frac{115 - 28}{5} = 17.4$

3rd $= a + 2d = 28 + 2 \times 17.4 = 62.8$ km/h

b) 1st $= a = 28$

6th $= ar^5 = 115$

$r^5 = \frac{115}{28}$

$r = \sqrt[5]{\frac{115}{28}} = 1.326502161$

5th $= ar^4 = 28 \times 1.326502161^4$

$= 86.6941671$

$= 86.7$ km/h^{-1} (1dp)

5. The value, £V, of a vintage car t years after it was first valued on 1st January 2001, is modelled by the equation

$$V = Ap^t \qquad \text{where } A \text{ and } p \text{ are constants}$$

Given that the value of the car was £32 000 on 1st January 2005 and £50 000 on 1st January 2012

$t=4$ $t=11$

(a) (i) find p to 4 decimal places,

(ii) show that A is approximately 24 800

(4)

(b) With reference to the model, interpret

(i) the value of the constant A,

(ii) the value of the constant p.

(2)

Using the model,

(c) find the year during which the value of the car first exceeds £100 000

(4)

(4)

a) (i) $V = Ap^t$

$32000 = Ap^4$ ①

$50000 = Ap^{11}$ ②

② ÷ ① $\dfrac{50000}{32000} = p^7$

$p = \sqrt[7]{\dfrac{50000}{32000}} = 1.0658$ (4dp)

(ii) ① gives $A = \dfrac{32000}{1.0658^4}$

$A = 24796.8022$

$A = 24800$ (3sf)

b) (i) A (£24800) is the value of the car on 1st January 2001

(ii) p is the multiplier indicating the percentage rise per year

$p = 1.0658$
is a rise of 6.58% per year

c) $Ap^t > 100000$

$24800 \times 1.0658^t > 100000$

$1.0658^t > \frac{100000}{24800}$

$\ln 1.0658^t > \ln\left(\frac{100000}{24800}\right)$

$t \ln 1.0658 > \ln \frac{100000}{24800}$

$t > \frac{\ln\left(\frac{100000}{24800}\right)}{\ln(1.0658)}$

$t > 21.8$

$2001 + 21.8 = 2022.8$

Year it first exceeds £100000 is 2022

7. (i) Show that $\sum_{r=1}^{16}(3 + 5r + 2^r) = 131\,798$

(4)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} u_r$

(3)

(i) $r=1 \quad 3+5\times1+2^1 = 10$

$r=2 \quad 3+5\times2+2^2 = 17$

$r=3 \quad 3+5\times3+2^3 = 26$

$r=4 \quad 3+5\times4+2^4 = 39$

$r=5 \quad 3+5\times5+2^5 = 60$

$r=6 \quad 3+5\times6+2^6 = 97$

$r=7 \quad 3+5\times7+2^7 = 166$

$r=8 \quad 3+5\times8+2^8 = 299$

$r=9 \quad 3+5\times9+2^9 = 560$

$r=10 \quad 3+5\times10+2^{10} = 1077$

$r=11 \quad 3+5\times11+2^{11} = 2106$

$r=12 \quad 3+5\times12+2^{12} = 4159$

$r=13 \quad 3+5\times13+2^{13} = 8260$

$r=14 \quad 3+5\times14+2^{14} = 16457$

$r=15 \quad 3+5\times15+2^{15} = 32846$

$r=16 \quad 3+5\times16+2^{16} = 65619$

Sum $= 131798$ as required

(ii) $u_{n+1} = \frac{1}{u_n} \quad u_1 = \frac{2}{3}$

$u_2 = \frac{1}{\frac{2}{3}} \quad \frac{3}{2}$

$u_3 = \frac{1}{\frac{3}{2}} = \frac{2}{3} \quad u^4 = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

alternates between $\frac{2}{3}$ and $\frac{3}{2}$

9. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$$

(3)

(i) $\displaystyle\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$

$= 20 \times \left(\frac{1}{2}\right)^4 + 20\left(\frac{1}{2}\right)^5 + 20\left(\frac{1}{2}\right)^6$

$= \frac{20}{16} + \frac{20}{16} \times \left(\frac{1}{2}\right) + \frac{20}{16}\left(\frac{1}{2}\right)^2 + \ldots$

$a = \frac{20}{16}$ $\quad r = \frac{1}{2}$ geometric progression

$S_\infty = \frac{a}{1-r} = \frac{\frac{20}{16}}{1-\frac{1}{2}} = \frac{5}{2}$

(ii) $\displaystyle\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right)$

$= \log_5\left(\frac{1+2}{1+1}\right) + \log_5\left(\frac{2+2}{2+1}\right) + \log_5\left(\frac{3+2}{3+1}\right)$

$\ldots + \log_5\left(\frac{47+2}{47+1}\right) + \log_5\left(\frac{48+2}{48+1}\right)$

$= \log_5\left(\frac{3}{2}\right) + \log_5\left(\frac{4}{3}\right) + \log_5\left(\frac{5}{4}\right)$

$+ \ldots \log_5\left(\frac{49}{48}\right) + \log_5\left(\frac{50}{49}\right)$

But, $\log_5 a + \log_5 b = \log_5(ab)$

$\therefore \log_5\left(\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \frac{6}{5} \times \ldots \frac{48}{47} \times \frac{49}{48} \times \frac{50}{49}\right)$

Then, cross cancelling the fractions

$$= \log_5\left(\frac{\cancel{3}}{2} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \frac{\cancel{6}}{\cancel{5}} \times \ldots \frac{\cancel{48}}{\cancel{47}} \times \frac{\cancel{49}}{\cancel{48}} \times \frac{50}{\cancel{49}}\right)$$

$$= \log_5\left(\frac{50}{2}\right)$$

$$= \log_5(25)$$

$$= 2 \quad \text{(as required)}$$

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

a) ① $S_n = a + ar + ar^2 + \dots + ar^{n-1}$

② $rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$

① − ② gives

$$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n)$$
$$S_n = \frac{a(1-r^n)}{1-r}$$

b) $S_{10} = 4S_5$

$$\frac{a(1-r^{10})}{1-r} = \frac{4a(1-r^5)}{1-r}$$

$$a - ar^{10} = 4a - 4ar^5$$
$$4\cancel{a}r^5 - \cancel{a}r^{10} = 3\cancel{a}$$
$$0 = r^{10} - 4r^5 + 3$$
$$0 = (r^5 - 3)(r^5 - 1)$$

Either $r^5 - 3 = 0$ or $r^5 - 1 = 0$

$r^5 = 3$ $\quad$ $r = 1$

$r = 3^{\frac{1}{5}} = \sqrt[5]{3}$ $\quad$ but $r \neq 1$

11. The function g is defined by

$$g(x) = \frac{3\ln(x) - 7}{\ln(x) - 2} \qquad x > 0 \qquad x \neq k$$

where k is a constant.

(a) Deduce the value of k.

(1)

(b) Prove that

$$g'(x) > 0$$

for all values of x in the domain of g.

(3)

(c) Find the range of values of a for which

$$g(a) > 0$$

(2)

a) $\ln(x) - 2 = 0$

$\ln x = 2$

$x = e^2$

$k = e^2$

b) Quotient rule

$u = 3\ln(x) - 7 \qquad v = \ln(x) - 2$

$u' = \frac{3}{x} \qquad v' = \frac{1}{x}$

$$g'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{-2 \times \frac{3}{x} - (-7) \times \frac{1}{x}}{(\ln(x) - 2)^2}$$

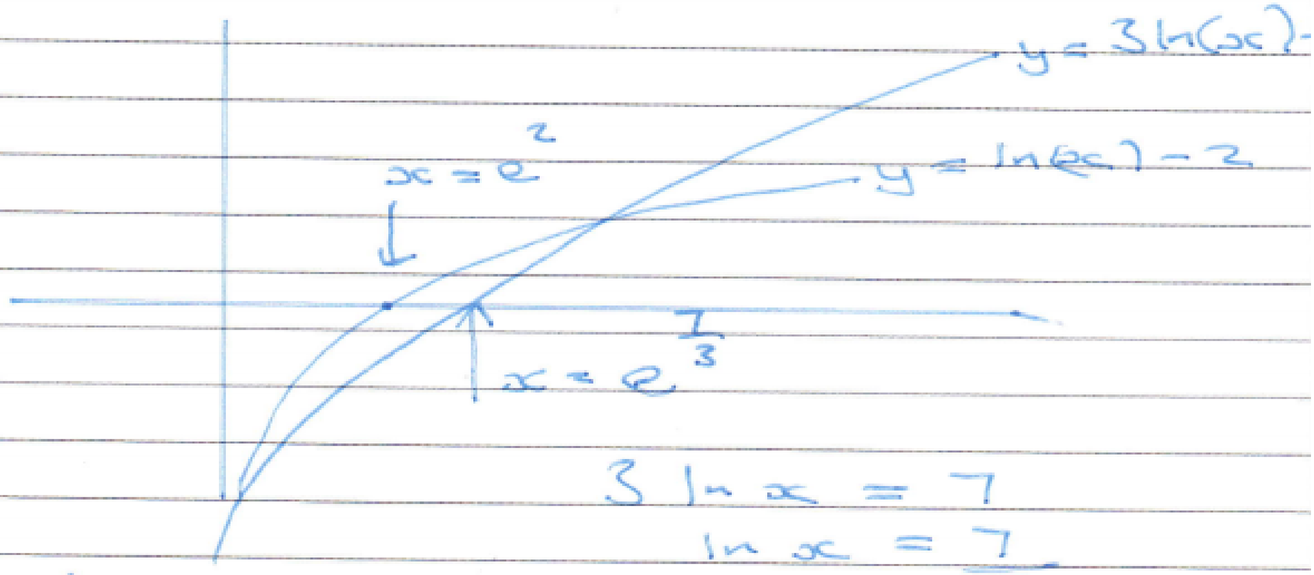
$$g'(x) = \frac{\frac{1}{x}}{(\ln(x) - 2)^2}$$

as $x > 0$, $\frac{1}{x}$ must be positive, and as denominator is squared, it must be positive, so $g'(x)$ must be > 0

c) $g(a) > 0$ when

$3\ln(x) - 7 > 0$ and $\ln(x) - 2 > 0$

or $3\ln(x) - 7 < 0$ and $\ln(x) - 2 < 0$



$3\ln x = 7$

$\ln x = \frac{7}{3}$

$x = e^{\frac{7}{3}}$

$\ln x = 2$

$x = e^2$

Range of values of a for $g(a) > 0$

$0 < a < e^2$ or $a > e^{\frac{7}{3}}$

(both < 0) (both > 0)