5.

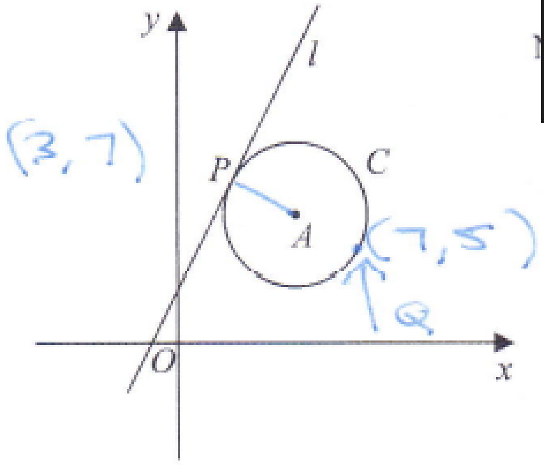


Figure 3

The circle C has centre A with coordinates (7, 5).

The line l, with equation $y = 2x + 1$, is the tangent to C at the point P, as shown in Figure 3.

(a) Show that an equation of the line PA is $2y + x = 17$

(3)

(b) Find an equation for C.

(4)

The line with equation $y = 2x + k$, $k \neq 1$ is also a tangent to C.

(c) Find the value of the constant k.

(3)

a) Gradient of tangent is 2

$\therefore$ gradient of radius PA $= -\frac{1}{2}$
(tangent and radius are perpendicular)

$m = -\frac{1}{2}$ $A(7,5)$

$y - y_1 = m(x - x_1)$
$y - 5 = -\frac{1}{2}(x - 7)$

$2y - 10 = -x + 7$
$2y + x = 10 + 7$
$2y + x = 17$ as required

b) find coordinates of P

tangent $y = 2x + 1$ ①
radius $2y + x = 17$ ②

sub ① in ②

$$2(2x+1) + x = 17$$
$$4x + 2 + x = 17$$
$$5x + 2 = 17$$
$$5x = 15$$
$$x = 3$$
$$\therefore \; y = 2\times 3 + 1 = 7$$

$P(3, 7)$

$$|AP| = \sqrt{(7-5)^2 + (3-7)^2} = 2\sqrt{5}$$

radius of circle $r = 2\sqrt{5}$
$\therefore r^2 = 20$

Equation of circle

$$(x-7)^2 + (y-5)^2 = 20$$

c) by symmetry Q (where 2nd tangent meets circle)

P $(3, 7)$
A $(7, 5)$ $\;\;+4,\ -2$
Q $(11, 3)$ $\;\;+4,\ -2$

$m = 2$
$Q(11, 3)$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - 11)$$
$$y - 3 = 2x - 22$$
$$y = 2x - 19 \quad \therefore k = -19$$

6. The function f is defined by

$$f(x) = \frac{3x-7}{x-2} \qquad x \in \mathbb{R}, x \neq 2$$

(a) Find $f^{-1}(7)$ **(2)**

(b) Show that $ff(x) = \dfrac{ax+b}{x-3}$ where a and b are integers to be found. **(3)**

a) $y = \dfrac{3x-7}{x-2}$

$y(x-2) = 3x-7$

$xy - 2y = 3x - 7$

$xy - 3x = 2y - 7$

$x(y-3) = 2y-7$

$x = \dfrac{2y-7}{y-3}$

$f^{-1}(x) = \dfrac{2x-7}{x-3}$

$f^{-1}(7) = \dfrac{2\times 7 - 7}{7-3} = \dfrac{7}{4}$

b) $ff(x) = \dfrac{3\left(\dfrac{3x-7}{x-2}\right) - 7}{\dfrac{3x-7}{x-2} - 2}$

$= \dfrac{\dfrac{9x-21}{x-2} - \dfrac{7(x-2)}{x-2}}{\dfrac{3x-7}{x-2} - \dfrac{2(x-2)}{x-2}}$

$= \dfrac{\dfrac{9x-21-7x+14}{x-2}}{\dfrac{3x-7-2x+4}{x-2}}$

$$= \frac{9x - 21 - 7x + 14}{3x - 7 - 2x + 4}$$

$$= \frac{2x - 7}{x - 3}$$

8. A curve C has parametric equations

$$x = 3 + 2\sin t, \quad y = 4 + 2\cos 2t, \quad 0 \leqslant t < 2\pi$$

(a) Show that all points on C satisfy $y = 6 - (x - 3)^2$

(2)

(b) (i) Sketch the curve C.

(ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2, \quad x \in \mathbb{R}$

(3)

a) $x = 3 + 2\sin t$ $\qquad y = 4 + 2\cos 2t$ (2)

$x - 3 = 2\sin t$

$\frac{x-3}{2} = \sin t$

$\left(\frac{x-3}{2}\right)^2 = \sin^2 t$

$\cos 2t = 1 - 2\sin^2 t$

$\cos 2t = 1 - 2\left(\frac{x-3}{2}\right)^2$ (1)

substitute (1) in (2)

$y = 4 + 2\left(1 - 2\left(\frac{x-3}{2}\right)^2\right)$

$y = 4 + 2 - \frac{4}{4}(x-3)^2$

$y = 6 - (x-3)^2$ as required

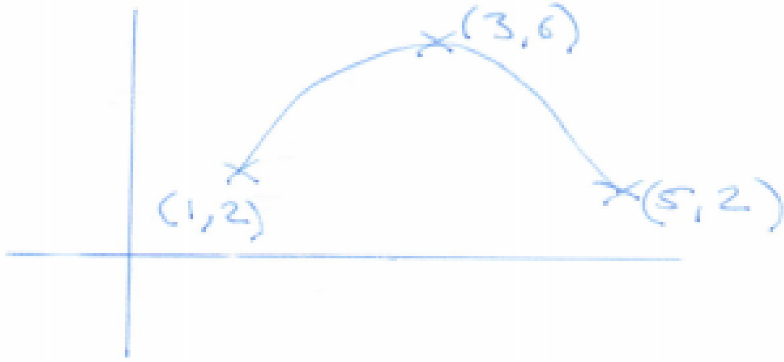
$0 \leq t < 2\pi$

Question 14 continued

b(i) $x = 3 + 2\sin t$

$$\boxed{\begin{aligned}&\text{max } x \text{ when } \sin t = 1 \ \left(t = \tfrac{\pi}{2}\right)\\ &x = 3 + 2\times 1 = 5,\ y = 4 + 2\cos\pi = 2\end{aligned}}$$

$$\boxed{\begin{aligned}&\text{min } x \text{ when } x = 3 + 2\times -1 = 1\\ &y = 4 + 2\cos\left(\tfrac{6\pi}{2}\right) = 2\end{aligned}}$$



(ii) $x = 3 + 2\sin t$, $1 \leq x \leq 5$ using parametric equation given above

$y = 6 - (x-3)^2 = 6 - (x^2 - 6x + 9)$

$= 6 - x^2 + 6x - 9$

$y = -x^2 + 6x - 3$

$\frac{dy}{dx} = -2x + 6$

$\frac{dy}{dx} = 0 \quad \therefore -2x + 6 = 0$

$x = 3$

when $x = 3$, $y = -9 + 18 - 3 = 6$

maximum point $(3,6)$

c) $x + y = k \Rightarrow y = k - x$

curve is $y = -x^2 + 6x - 3$

$\therefore \quad k - x = -x^2 + 6x - 3$

$x^2 - 6x - x + k + 3 = 0$

$x^2 - 7x + (k+3) = 0$