2.

$$g(x) = \frac{2x+5}{x-3} \qquad x \geqslant 5$$

(a) Find gg(5).

(2)

(b) State the range of g.

(1)

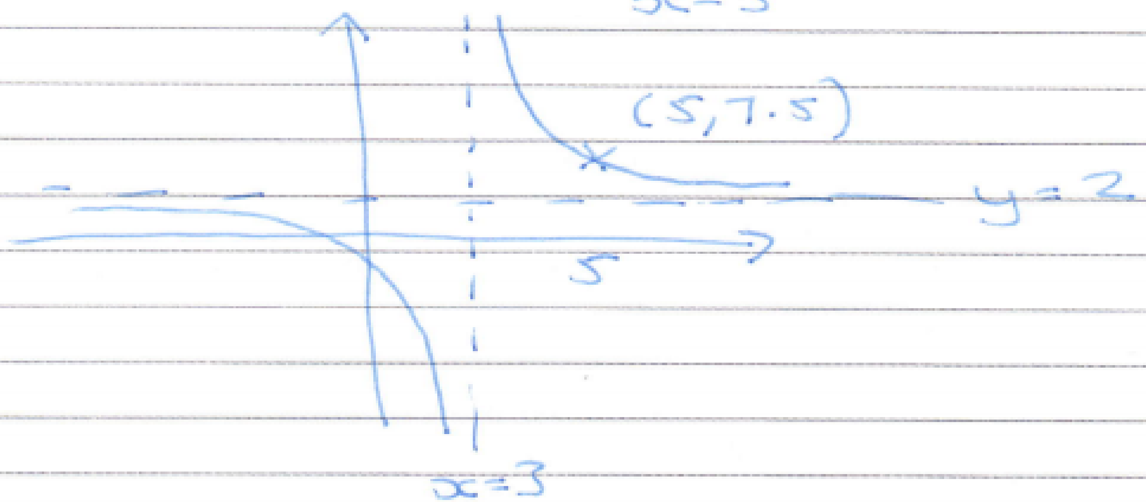
(c) Find $g^{-1}(x)$, stating its domain.

(3)

a) $g(5) = \frac{2 \times 5 + 5}{5 - 3} = 7.5$

$g(7.5) = \frac{2 \times 7.5 + 5}{7.5 - 3} = \frac{40}{9}$

b) graph $g(x) = \frac{2x+5}{x-3}$



So for $x \geqslant 5$
max point $(5, 7.5)$
asymptote at $y = 2$

range $2 < g(x) \leq 7.5$

c) $y = \dfrac{2x+5}{x-3}$ $x \geq 5$

$y(x-3) = 2x+5$

$xy - 3y = 2x + 5$

$xy - 2x = 3y + 5$

$x(y-2) = 3y + 5$

$x = \dfrac{3y+5}{y-2}$

$g^{-1}(x) = \dfrac{3x+5}{x-2}$

domain is range of $g(x)$

$2 < x \leq 7.5$

4. The curve with equation $y = 3 \times 2^x$ meets the curve with equation $y = 15 - 2^{x+1}$ at the point P.

Find, using algebra, the exact x coordinate of P.

(4)

$3 \times 2^x = 15 - 2^{x+1}$

$3 \times 2^x = 15 - 2^x \times 2^1$

$3 \times 2^x + 2 \times 2^x = 15$

$5 \times 2^x = 15$

$2^x = \frac{15}{5}$

$2^x = 3$

$\log_2 2^x = \log_2 3$

$x = \log_2 3$

5. A curve C has equation $y = f(x)$

Given that

- $f'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $f(x)$

find, in simplest form, $f(x)$

(6)

$f'(x) = 6x^2 + ax - 23$

$f(x) = 2x^3 + \frac{1}{2}ax^2 - 23x + c$

at y intercept -12

$-12 = c$

$f(x) = 2x^3 + \frac{1}{2}ax^2 - 23x - 12$

if $(x+4)$ is a factor, $f(-4) = 0$

$2(-4)^3 + \frac{1}{2} \times a(-4)^2 - 23(-4) - 12 = 0$

$-128 + 8a + 92 - 12 = 0$

$8a = 48$

$a = 6$

$f(x) = x^3 + 3x^2 - 23x - 12$

6.

$$f(x) = -3x^3 + 8x^2 - 9x + 10, \quad x \in \mathbb{R}$$

(a) (i) Calculate f(2)

(ii) Write f(x) as a product of two algebraic factors.

(3)

Using the answer to (a)(ii),

(b) prove that there are exactly two real solutions to the equation

$$-3y^6 + 8y^4 - 9y^2 + 10 = 0$$

(2)

(c) deduce the number of real solutions, for $7\pi \leqslant \theta < 10\pi$, to the equation

$$3\tan^3\theta - 8\tan^2\theta + 9\tan\theta - 10 = 0$$

(1)

a (i) $f(2) = -3(2)^3 + 8(2)^2 - 9(2) + 10$

$= 0$

(ii) $\therefore$ $x - 2$ is a factor

$$\begin{array}{r|l} & -3x^2 + 2x - 5 \\ x - 2 & -3x^3 + 8x^2 - 9x + 10 \\ - & -3x^3 + 6x^2 \\ & 2x^2 - 9x \\ - & 2x^2 - 4x \\ & -5x + 10 \\ - & -5x + 10 \\ & 0 \end{array}$$

$f(x) = (x-2)(-3x^2 + 2x - 5)$

b) $-3y^6 + 8y^4 - 9y^2 + 10 = 0$

Let $x = y^2$

$-3x^3 + 8x^2 - 9x + 10 = 0$

$(x-2)(-3x^2 + 2x - 5) = 0$

7.

$$f(x) = 2x^2 + 4x + 9 \qquad x \in \mathbb{R}$$

(a) Write f(x) in the form $a(x + b)^2 + c$, where a, b and c are integers to be found.

(3)

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

(3)

(c) (i) Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where

$$g(x) = 2(x - 2)^2 + 4x - 3 \qquad x \in \mathbb{R}$$

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \qquad x \in \mathbb{R}$$

(4)

a) $2x^2 + 4x + 9$

$= 2\left(x^2 + 2x + \frac{9}{2}\right)$

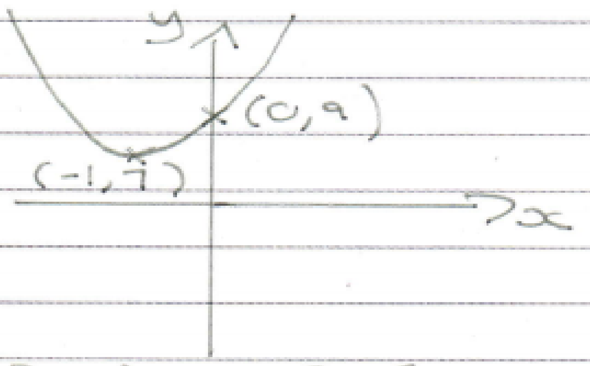
$= 2\left((x+1)^2 - 1 + \frac{9}{2}\right)$

$= 2\left((x+1)^2 + \frac{7}{2}\right)$ $\qquad a = 2$

$\qquad b = 1$

$= 2(x+1)^2 + 7$ $\qquad c = 7$

b)



at $x = 0, y = 9$

minimum pt $(-1, 7)$

c) $f(x) = 2x^2 + 4x + 9$

shift 2 right $f(x-2)$

$f(x-2) = 2(x-2)^2 + 4(x-2) + 9$

$= 2(x-2)^2 + 4x - 8 + 9$

$= 2(x-2)^2 + 4x + 1$

Then to transform to $g(x)$ need -4

Full transformation is

translate $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

5c (ii) $f(x) = 2(x+1)^2 + 7$

minimum point $= (-1, 7)$

$h(x) = \dfrac{21}{2x^2+4x+9} = 21(2x^2+4x+9)^{-1}$

$h'(x) = -21(2x^2+4x+9)^{-2} \times (4x+4)$

$= \dfrac{-21(4x+4)}{(2x^2+4x+9)^2}$

$h'(x) = 0$ when $x = -1$

$h(x) = \dfrac{21}{2(-1)^2+4(-1)+9} = \dfrac{21}{7} = 3$

as $x \to \infty$ or $x \to -\infty$

$h(x) \to 0$

$\therefore$ range $0 < h(x) \leq 3$

8. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre. After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, **(2)**

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. **(4)**

a) 1st 2nd 3rd 4th 5th 6th

6 6 6 6 6.3 6.615

$\times 1.05$ $\times 1.05$

$\therefore$ 6 + 6 + 6 + 6 + 6.3 + 6.615

= 36.915 minutes

0.915 × 60 = 54.9 seconds

$\therefore$ Time for 1st 6 km = 36 mins 55 seconds (to nearest second)

b) 4th 5th 6th

6 6.3 6.615

$\times 1.05$ $\times 1.05$

Geometric progression

a = 6 (4th term)

common ratio = 1.05

r = 4th term = 6

r = 5th term = 6×1.05^1 → 5 − 4

r = 6th term = 6×1.05^2 → 6 − 4

$\therefore$ rth term = $6 \times 1.05^{r-4}$

11 c) 1 1km 6 mins

2 1km 6 mins

3 1km 6 mins

---

4 1km 6 mins

5 1km 6.3 mins = 6 × 1.05

a r

Geometric progression

from 4 up to 20 km

$$S_n = \frac{a(1-r^n)}{1-r}$$

a = 6

r = 1.05

n = 17

$$S_{17} = \frac{6(1-1.05^{17})}{1-1.05}$$

$= 155.024981$ minutes

$+ 18$ minutes

$= 173.042981$

$0.042981 \times 60 = 2.531886 = 3$ secs

Estimated time = 173 minutes 3 seconds

11.

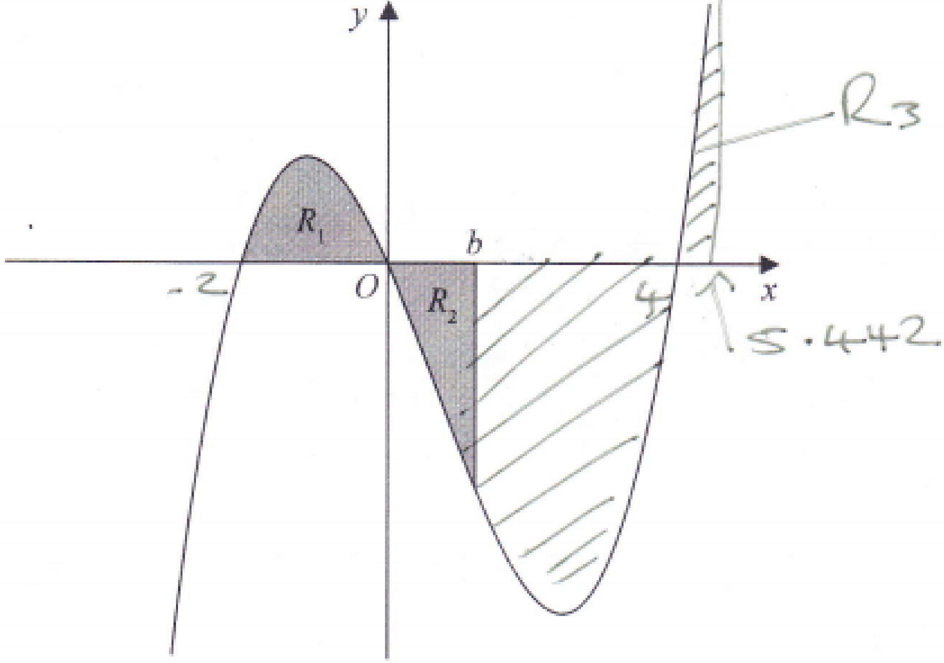


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = x(x + 2)(x - 4)$.

The region R_1 shown shaded in Figure 2 is bounded by the curve and the negative x-axis.

(a) Show that the exact area of R_1 is $\frac{20}{3}$

(4)

The region R_2 also shown shaded in Figure 2 is bounded by the curve, the positive x-axis and the line with equation $x = b$, where b is a positive constant and $0 < b < 4$

Given that the area of R_1 is equal to the area of R_2

(b) verify that b satisfies the equation

$$(b + 2)^2 (3b^2 - 20b + 20) = 0$$

(4)

The roots of the equation $3b^2 - 20b + 20 = 0$ are 1.225 and 5.442 to 3 decimal places. The value of b is therefore 1.225 to 3 decimal places.

(c) Explain, with the aid of a diagram, the significance of the root 5.442

(2)

(2)

a) $y = x(x^2 + 2x - 4x - 8)$

$= x(x^2 - 2x - 8)$

$= x^3 - 2x^2 - 8x$

$R_1 \int_{-2}^{0} (x^3 - 2x^2 - 8x)\,dx = \left[\frac{x^4}{4} - \frac{2}{3}x^3 - 4x^2\right]_{-2}^{0}$

$= (0 - 0 - 0) - \left(\frac{16}{4} - \frac{2}{3}(-2)^3 - 4(-2)^2\right) = \frac{20}{3}$

$$\int_0^b (x^3 - 2x^2 - 8x)\,dx = -\frac{20}{3}$$ under the line

$$\left[\frac{x^4}{4} - \frac{2}{3}x^3 - 4x^2\right]_0^b = -\frac{20}{3}$$

$$\left(\frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2\right) - (0 - 0 - 0) = -\frac{20}{3}$$

$$\frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2 = -\frac{20}{3}$$

$$\frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2 + \frac{20}{3} = 0$$

$\times$ through by 12

$$12 \times \frac{b^4}{4} - 12 \times \frac{2}{3}b^3 - 12 \times 4b^2 + 12 \times \frac{20}{3} = 0$$

$$3b^4 - 8b^3 - 48b^2 + 80 = 0 \quad (1)$$

Verify that (1) same as

$$(b+2)^2(3b^2 - 20b + 20) = 0$$

$$(b+2)\left[3b^3 - 20b^2 + 20b + 6b^2 - 40b + 40\right] = 0$$

$$(b+2)(3b^3 - 14b^2 - 20b + 40) = 0$$

$$3b^4 - 14b^3 - 20b^2 + 40b + 6b^3 - 28b^2$$
$$- 40b + 80 = 0$$

$$3b^4 - 8b^3 - 48b^2 + 80 = 0 \quad (2)$$

as (2) = (1)

b satisfies $(b+2)^2(3b^2 - 20b + 20) = 0$

c) Area shaded on graph between -2 and 5.442 – the area shaded above the x-axis is the same as the area shaded below the x-axis between -2 and 5.442