2 A curve has equation $y = x^5 + 4x^3 + 7x + q$ where q is a positive constant.

Find the gradient of the curve at the point where $x = 0$

Circle your answer.

$\frac{dy}{dx} = 5x^4 + 12x^2 + 7$

[1 mark]

0 4 7 q

13 A curve, C, has equation

$$y = \frac{e^{3x-5}}{x^2}$$ — u ← v

Show that C has exactly one stationary point.

Fully justify your answer.

[7 marks]

quotient rule

$u = e^{3x-5}$ $v = x^2$

$\frac{du}{dx} = 3e^{3x-5}$ $\frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$0 = \frac{3x^2 e^{3x-5} - 2x e^{3x-5}}{x^4}$

$\frac{dy}{dx} = 0$ for a stationary point

$0 = \frac{x e^{3x-5}(3x-2)}{x^4}$

e^{3x-5} cannot be zero

x cannot be zero as we cannot divide by zero

$\therefore$ only solution is $3x - 2 = 0$

$x = \frac{2}{3}$

15 A curve has equation $y = x^3 - 48x$

The point A on the curve has x coordinate -4

The point B on the curve has x coordinate $-4 + h$

15 (a) Show that the gradient of the line AB is $h^2 - 12h$

[4 marks]

$y = x^3 - 48x$

$\frac{dy}{dx} = 3x^2 - 48$

at A, $x = 4$, $y = (-4)^3 - 48 \times (-4) = 128$

B, $x = (-4+h)$, $y = (-4+h)^3 - 48(-4+h)$

$y = (-4+h)(16 - 8h + h^2) + 192 - 48h$

$= -64 + 32h - 4h^2$

$+ 16h - 8h^2 + h^3 + 192 - 48h$

$y = h^3 - 12h^2 + 128$

$A(-4, -128)$

$B(-4+h, h^3 - 12h^2 + 128)$

gradient $= \frac{(h^3 - 12h^2 + 128) - 128}{h}$

$= \frac{h^3 - 12h^2}{h} = h^2 - 12h$ (as required)

15 (b) Explain how the result of part **(a)** can be used to show that A is a stationary point on the curve.

[2 marks]

at A, $\frac{dy}{dx} = 4 \times (-4)^2 - 48$

$= 0$

as $\frac{dy}{dx} = 0$ at A it is a stationary point.

10 The volume of a spherical bubble is increasing at a constant rate. $V = kt$

Show that the rate of increase of the radius, r, of the bubble is inversely proportional to r^2

$\frac{dr}{dt}$

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

[4 marks]

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ (differentiating)

1 $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

if $V = kt$

$\frac{dV}{dt} = k$

substituting in 1

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times k$

$\therefore \frac{dr}{dt} \propto \frac{1}{r^2}$

Do not write outside the box

12 A curve C has equation

$$x^3 \sin y + \cos y = Ax$$

where A is a constant.

C passes through the point $P\left(\sqrt{3}, \frac{\pi}{6}\right)$

12 (a) Show that $A = 2$

[2 marks]

$x = \sqrt{3}$, $y = \frac{\pi}{6}$

$(\sqrt{3})^3 \times \sin\frac{\pi}{6} + \cos\frac{\pi}{6} = A \times \sqrt{3}$

$\frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = A\sqrt{3}$

$\frac{4\sqrt{3}}{2} = A\sqrt{3}$

$2\sqrt{3} = A\sqrt{3} \quad \therefore A = 2$

12 (b) (i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - 3x^2 \sin y}{x^3 \cos y - \sin y}$

[5 marks]

$x^3 \sin y + \cos y - 2x = 0$

implicit differentiation

$3x^2 \sin y + x^3 \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} - 2 = 0$

$x^3 \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} = 2 - 3x^2 \sin y$

$\frac{dy}{dx}(x^3 \cos y - \sin y) = 2 - 3x^2 \sin y$

$\frac{dy}{dx} = \frac{2 - 3x^2 \sin y}{x^3 \cos y - \sin y}$

12 (b) (ii) Hence, find the gradient of the curve at P.

[2 marks]

$$\frac{dy}{dx} = \frac{2 - 3 \times 3 \times \sin\frac{\pi}{6}}{3\sqrt{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{6}}$$

$$\frac{dy}{dx} = -\frac{5}{8}$$

12 (b) (iii) The tangent to C at P intersects the x-axis at Q. $y = 0$

Find the exact x-coordinate of Q.

[4 marks]

$m = -\frac{5}{8}$ $\left(\sqrt{3}, \frac{\pi}{6}\right)$

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{6} = -\frac{5}{8}(x - \sqrt{3})$

When $y = 0$ meets x-axis

$-\frac{\pi}{6} \times \frac{8}{5} = x - \sqrt{3}$

$\frac{8}{30}\pi + \sqrt{3} = x$

$\frac{4}{15}\pi + \sqrt{3} = x$

outside the box

6 A function f is defined by $f(x) = \frac{x}{\sqrt{2x-2}}$

6 (a) State the maximum possible domain of f.

[2 marks]

Cannot divide by 0

$\therefore\ 2x - 2 > 0$

$2x > 2$

$x > 1$

$\{x \in \mathbb{R} : x > 1\}$

6 (b) Use the quotient rule to show that $f'(x) = \frac{x-2}{(2x-2)^{\frac{3}{2}}}$

[3 marks]

$u = x \qquad v = (2x-2)^{\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{2}{2}(2x-2)^{-\frac{1}{2}}$

$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x-2)^{\frac{1}{2}} - x(2x-2)^{-\frac{1}{2}}}{2x-2}$

$= \frac{(2x-2)^{-\frac{1}{2}}[(2x-2) - x]}{(2x-2)}$

$= \frac{x-2}{(2x-2)^{\frac{1}{2}}(2x-2)}$

$= \frac{x-2}{(2x-2)^{\frac{3}{2}}}$ (as required)

6 (c) Show that the graph of $y = f(x)$ has exactly one point of inflection.

[7 marks]

For point of inflection

$f''(x) = 0$

$u = x - 2 \qquad v = (2x-2)^{\frac{3}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{3}{2} \times 2(2x-2)^{\frac{1}{2}}$

$$f''(x) = \frac{(2x-2)^{\frac{3}{2}} - 3(x-2)(2x-2)^{\frac{1}{2}}}{(2x-2)^3}$$

if $f''(x) = 0$

$0 = (2x-2)^{\frac{1}{2}}\left[(2x-2) - 3(x-2)\right]$

$0 = (2x-2)^{\frac{1}{2}}\left[2x - 2 - 3x + 6\right]$

$0 = (2x-2)^{\frac{1}{2}}(-x+4)$

Either $x = 1$ or $x = 4$

from a) domain is $x > 1$

$\therefore$ only $x = 4$ is a point of inflection

$f''(3) = \frac{1}{32} > 0$ convex

$f''(5) = \frac{-\sqrt{2}}{256} < 0$ concave

6 (d) Write down the values of x for which the graph of $y = f(x)$ is convex.

[1 mark]

$1 < x < 4$