YORKSHIRE MATHS TUTOR

Practice Paper F

1. $$\frac{6x+42}{(5x-1)(2x+5)} = \frac{A}{5x-1} + \frac{B}{2x+5}$$

$$6x+42 = A(2x+5) + B(5x-1)$$

let $x = -\frac{5}{2}$ $\quad 6\times -\frac{5}{2} + 42 = B\left(5\times -\frac{5}{2} - 1\right)$

$$27 = -\frac{27}{2}B$$

$$B = \frac{2\times 27}{-27} = -2$$

let $x = \frac{1}{5}$ $\quad 6\times\frac{1}{5} + 42 = A\left(2\times\frac{1}{5}+5\right)$

$$\frac{216}{\cancel{5}} = \frac{27}{\cancel{5}}A$$

$$A = \frac{216}{27} = 8$$

$A = 8$, $B = -2$

2. Assume there exist integers a and b such that $25a + 15b = 1$

$\div$ through by 5

$$5a + 3b = \frac{1}{5}$$

if a is an integer $5a$ is integer
if b is an integer $3b$ is integer

Adding two integers cannot give $\frac{1}{5}$ so this is a contradiction.
Therefore there do not exist integers a and b such that $25a + 15b = 1$

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3. $x = \cos 2t \qquad y = \sin t$

a) $\frac{dx}{dt} = -2\sin 2t \qquad \frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \frac{\cos t}{-2\sin 2t} = \frac{\cancel{\cos t}^{\,1}}{-2\times 2\sin t\cancel{\cos t}_{\,1}}$

$= -\frac{1}{4\sin t} = -\frac{1}{4}\operatorname{cosec} t$

b) at $t = -\frac{5\pi}{6}$, $\frac{dy}{dx} = -\frac{1}{4\sin\left(-\frac{5\pi}{6}\right)}$

$\frac{dy}{dx} = \frac{1}{2}$

$\therefore$ gradient of normal is -2

at $t = -\frac{5\pi}{6}$, $x = \cos\left(2\times -\frac{5\pi}{6}\right)$

$x = \frac{1}{2}$

at $t = -\frac{5\pi}{6}$, $y = \sin\left(-\frac{5\pi}{6}\right)$

$y = -\frac{1}{2}$

$m = -2$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$

$y - y_1 = m(x - x_1)$

$y + \frac{1}{2} = -2\left(x - \frac{1}{2}\right)$

$y + \frac{1}{2} = -2x + 1$

$y = -2x + \frac{1}{2}$

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4. $I = \int \cot 3x \, dx$

$$\cot 3x = \frac{\cos 3x}{\sin 3x}$$

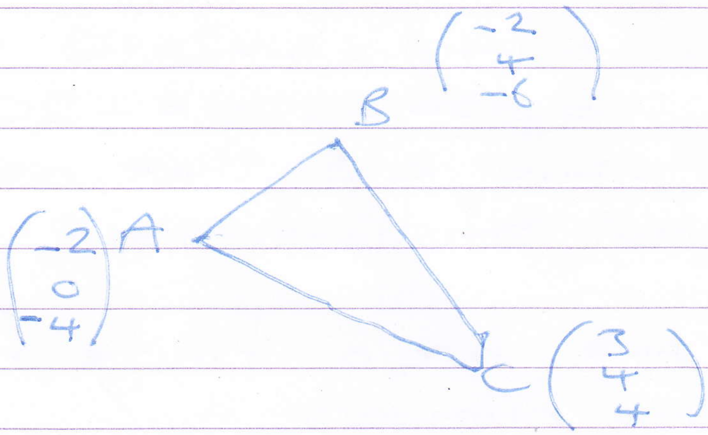
Let $y = \ln|\sin 3x|$

$$\frac{dy}{dx} = \frac{3\cos 3x}{\sin 3x} = 3\cot 3x$$

Using reverse chain rule

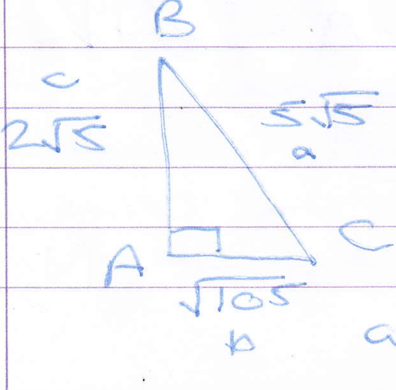
$$I = \frac{1}{3} \ln \sin |\sin 3x| + c$$

5.



$$AB = \sqrt{(-2--2)^2 + (4-0)^2 + (-6+4)^2} = 2\sqrt{5}$$
$$BC = \sqrt{(3--2)^2 + (4-4)^2 + (4--6)^2} = 5\sqrt{5}$$
$$AC = \sqrt{(3--2)^2 + (4-0)^2 + (4+4)^2} = \sqrt{105}$$

BC is largest length



Pythagoras $a^2 = b^2 + c^2$

$$b^2 + c^2 = (2\sqrt{5})^2 + (\sqrt{105})^2$$
$$= 20 + 105 = 125$$
$$a^2 = (5\sqrt{5})^2 = 125$$

as $a^2 = b^2 + c^2$ triangle is right angled

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6. $p : x \rightarrow x^2 \qquad q : x \rightarrow 5 - 2x$

a) $pq(x) = (5-2x)^2$

$qp(x) = 5 - 2x^2$

if $pq(x) = qp(x)$

$(5-2x)^2 = 5 - 2x^2$

$25 - 20x + 4x^2 = 5 - 2x^2$

$4x^2 + 2x^2 - 20x + 25 - 5 = 0$

$6x^2 - 20x + 20 = 0$

$3x^2 - 10x + 10 = 0$

b) $a = 3 \qquad b = -10 \qquad c = 10$

$b^2 - 4ac = (-10)^2 - 4 \times 3 \times 10$

$= 100 - 120 = -20$

as $b^2 - 4ac < 0$ there are no real solutions

7. Assume there are a finite number of prime numbers

Let primes be $p_1, p_2, p_3 \ldots p_n$

Consider a number one greater than the product of existing primes

Let $N = (p_1 \times p_2 \times p_3 \ldots \times p_n) + 1$

If we divide N by any existing prime we will get remainder of 1

So none of the existing prime numbers is a factor of N

This means N must be prime or N has a prime factor not currently listed.

Either way, this is a contradiction, so therefore there is an infinite number of prime numbers.

8. $\frac{dF}{dt} \propto -F$

$\frac{dF}{dt} = -kF$

9. Month 1 2 3 n

100 105

5% increase $\times 1.05$ $\times 1.05$

$a = 100$ $r = 1.05$

a) $S_n = \frac{a(1-r^n)}{1-r}$

$S_9 = \frac{100(1-1.05^9)}{1-1.05} = £1102.66$

b) $\frac{100(1-1.05^n)}{1-1.05} > 6000$

$\frac{100}{-0.05}(1-1.05^n) > 6000$

$-2000(1-1.05^n) > 6000$

$-(1-1.05^n) > \frac{6000}{2000}$

$-1 + 1.05^n > 3$

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$$1.05^n > 4$$
$$\ln 1.05^n > \ln 4$$
$$n \ln 1.05 > \ln 4$$
$$n > \frac{\ln 4}{\ln 1.05}$$

9c) $a = 50$ $d = ?$ $n = 29$

Arithmetic series $S_n = \frac{n}{2}(2a + (n-1)d)$

$$S_{29} = \frac{29}{2}(100 + 28d) = 6000$$

$$14.5(100 + 28d) = 6000$$
$$1450 + 406d = 6000$$
$$406d = 6000 - 1450$$
$$406d = 4550$$
$d = £11.21$ (nearest penny)

10. $\int \cos^2 6x \, dx$

Identities
$$\cos 2x = 2\cos^2 x - 1$$
$$\frac{\cos 2x + 1}{2} = \cos^2 x$$

replace x with $2x$

$$\frac{\cos 12x}{2} + \frac{1}{2} = \cos^2 6x$$

$$\therefore \int \frac{1}{2}\cos 12x + \frac{1}{2} \, dx$$

$$= \frac{1}{24}\sin 12x + \frac{1}{2}x + c$$

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7/

1 (a) $\dfrac{\tan x - \sec x}{1 - \sin x} = \dfrac{\dfrac{\sin x}{\cos x} - \dfrac{1}{\cos x}}{1 - \sin x}$

$= \dfrac{\sin x - 1}{\cos x} \div (1 - \sin x)$

$= \dfrac{\sin x - 1}{\cos x} \times \dfrac{1}{1 - \sin x}$

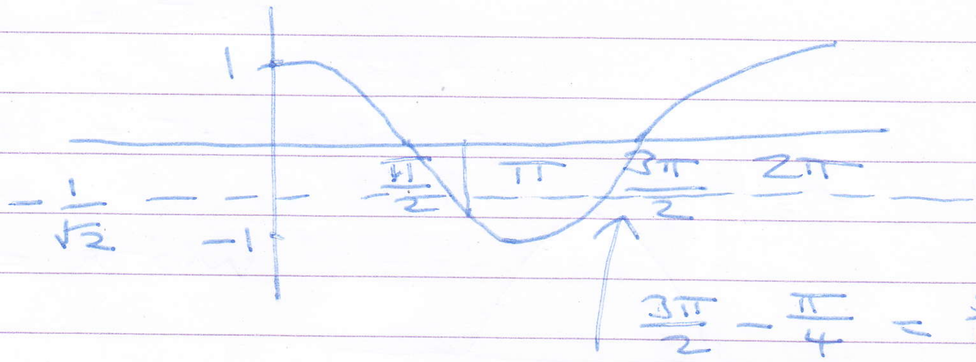
$= \dfrac{-\cancel{(1 - \sin x)}}{\cos x} \times \dfrac{1}{\cancel{(1 - \sin x)}} = \dfrac{-1}{\cos x}$

$\equiv -\sec x$ (as required)

b) $-\dfrac{1}{\cos x} = \sqrt{2}$ $\quad 0 \leq x \leq 2\pi$

$\cos x = -\dfrac{1}{\sqrt{2}}$

$x = \cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right) = \dfrac{3}{4}\pi$



$\dfrac{3\pi}{2} - \dfrac{\pi}{4} = \dfrac{5}{4}\pi$

$x = \dfrac{3}{4}\pi, \dfrac{5}{4}\pi$

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12. $x = 8t + 80 \qquad y = 100 - t^2$

a) $\frac{x-80}{8} = t$ sub in equation for y

$y = 100 - \left(\frac{x-80}{8}\right)^2$

$y = 100 - \left(\frac{x^2 - 160x + 6400}{64}\right)$

$y = 100 - \frac{1}{64}x^2 + \frac{5}{2}x - 100$

$y = -\frac{1}{64}x^2 + \frac{5}{2}x$

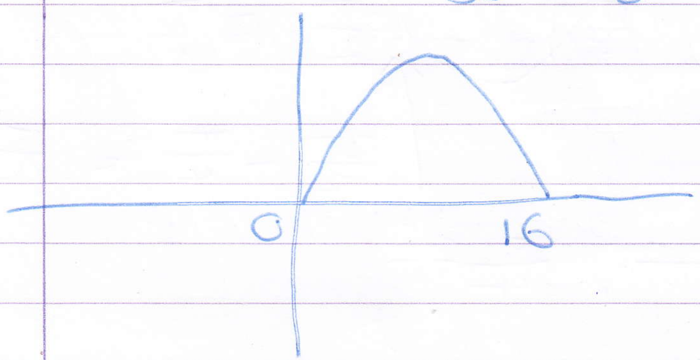
b) width $\rightarrow$ where $-\frac{1}{64}x^2 + \frac{5}{2}x = 0$

$x\left(-\frac{1}{64}x + \frac{5}{2}\right) = 0$

$x = 0$ and $-\frac{1}{64}x + \frac{5}{2} = 0$

$2.5 = \frac{1}{64}x$

$x = 64 \times 2.5 = 160$



width $= 160$ m

c) Height $y = -\frac{1}{64}x^2 + 2.5x$

$\frac{dy}{dx} = -\frac{1}{32}x + 2.5$

Max $0 = -\frac{1}{32}x + 2.5$

$\frac{1}{32x} = 2.5 \Rightarrow x = \frac{1}{32 \times 2.5} = \frac{1}{80}$

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$$y = -\frac{1}{64}(80^2) + 2.5 \times 80$$

$y = 100$ m is greatest height

13. $$\frac{x^3 + 8x^2 - 9x + 12}{x + 6}$$

$$\begin{array}{r}
x^2 + 2x - 21 \\
x + 6 \overline{) x^3 + 8x^2 - 9x + 12} \\
- \underline{x^3 + 6x^2} \\
2x^2 - 9x \\
- \underline{2x^2 + 12x} \\
-21x + 12 \\
- \underline{-21x - 126} \\
138
\end{array}$$

$A = 1,\ B = 2,\ C = -21,\ C = 138$

14. $V = \frac{4}{3}\pi r^3$ $\quad S = 4\pi r^2$

Variables V S r t

$\frac{dV}{dr} = 4\pi r^2$ $\quad$ differentiating

$\frac{dS}{dr} = 8\pi r$ $\quad \frac{dS}{dt} = -12$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dS} \times \frac{dS}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{8\pi r} \times -12$$

$$\frac{dV}{dt} = -6r$$

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15. $$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \times \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$$

if $I = \int \sin x \cos^2 x \, dx$

let $y = \cos^3 x$

$$\frac{dy}{dx} = -3\cos^2 x \sin x$$

$$\therefore I = -\frac{1}{3}\cos^3 x$$

$$\therefore \int \sin x \, dx - \int \sin x \cos^2 x \, dx$$

$$= -\cos x + \frac{1}{3}\cos^3 x + c$$

16. a) $h(19.3) = 40\ln(19.3+1) + 40\sin\left(\frac{19.3}{5}\right) - \frac{1}{4}(19.3)^2 = 0.97487$

$h(19.4) = 40\ln(19.4+1) + 40\sin\left(\frac{19.4}{5}\right) - \frac{1}{4}(19.4)^2 = -0.3934$

as there is a chage of sign between $h(19.3)$ and $h(19.4)$ it returns to the ground betwen $t = 19.3$ and $t = 19.4$

16 b) $h(t) = 40\ln(t+1) + 40\sin\left(\frac{t}{5}\right) - \frac{1}{4}t^2$

$$x_1 = x_0 - \frac{h(x_n)}{h'(x_n)}$$

$h(t_0) = 40\ln(19.35+1) + 40\sin\left(\frac{19.35}{5}\right) - \frac{1}{4}\times(19.35)^2 = 0.29033207$

$h'(t) = \frac{40}{t+1} + 40\times\frac{1}{5}\cos\left(\frac{t}{5}\right) - \frac{1}{2}t$

at $t = 19.35$

$h'(t) = \frac{40}{19.35+1} + 8\cos(19.35) - \frac{1}{2}\times 19.35$

$= -13.67928243$

$x_1 = 19.35 - \left(\frac{0.29033207}{-13.67928243}\right)$

$x_1 = 19.371$ (3dp)

c) $h(19.3705) = 0.0100055$

$h(19.3715) = -3.6638\times10^{-3}$

As there is a change of sign there is a root between 19.3705 and 19.3715 so 19.371 is a root to 3dp

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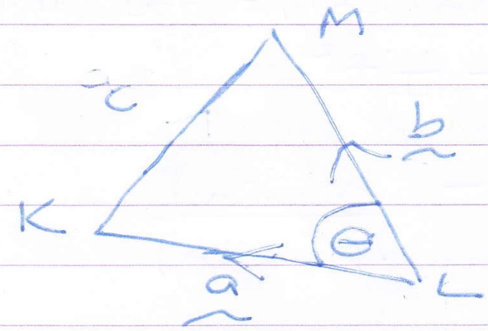
17. a)



arrows pointing away from angle θ

$$\underset{\sim}{a} = \overrightarrow{LK} = -\overrightarrow{KL} = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$$

$$\underset{\sim}{b} = \overrightarrow{LM} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$$

$$\cos\theta = \frac{\begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}}{\sqrt{(-3)^2+0^2+6^2} \times \sqrt{2^2+5^2+4^2}}$$

$$\cos\theta = \frac{-6+0+24}{\sqrt{45} \times \sqrt{45}}$$

$$\cos\theta = \frac{18}{45}$$

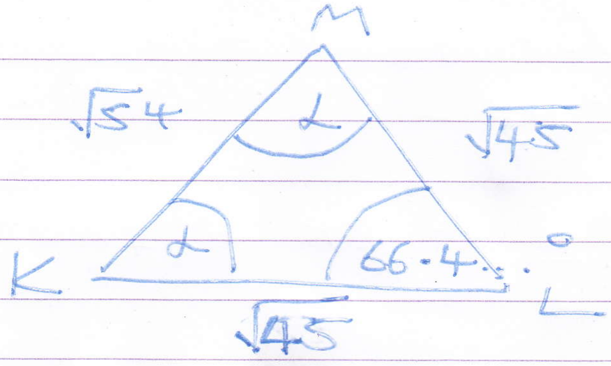
$$\theta = \cos^{-1}\frac{18}{45} = 66.4^\circ \text{ (1dp)}$$

b) Use cosine rule to find $|\overrightarrow{KM}|$

$$c^2 = (\sqrt{45})^2 + (\sqrt{45})^2 - 2 \times \sqrt{45} \times \sqrt{45} \times \cos(66.4)$$

$$c^2 = 54$$

$$|\overrightarrow{KM}| = c = \sqrt{54}$$



From diagram $\triangle KLM$ is isosceles

$\therefore \quad \alpha = (180 - 66.42182) \div 2$

$= 56.789$

$= 56.8°$ (1dp)

$\angle LKM = \angle LMK = 56.8°$