10. A geometric series is $a + ar + ar^2 + ...$

(a) Prove that the sum of the first n terms of this series is given by

$$S_n = \frac{a(1-r^n)}{1-r}.$$

(4)

(b) Find

$$\sum_{k=1}^{10} 100(2^k).$$

(3)

(c) Find the sum to infinity of the geometric series

$$\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + ...$$

(3)

(d) State the condition for an infinite geometric series with common ratio r to be convergent.

(1)

a) $S_n = a + ar + ar^2 + ... ar^{n-1}$ (1)

$rS_n = ar + ar^2 + ar^3 + ... ar^n$ (2)

Equation (1) − (2)

$rS_n - S_n = ar^n - a$

$S_n(r-1) = a(r^n - 1)$

× each side by −1

$-S_n(r-1) = -a(r^n-1)$

$S_n(1-r) = a(1-r^n)$

$S_n = \dfrac{a(1-r^n)}{1-r}$

b) $a = 200,$

$r = 2$

$n = 2$

$k=1, \sum_{k=1}^{10} = 100(2^1) = 200$

$k=2 \sum_{k=1}^{10} 100(2^2) = 400$

$k=3 \sum_{k=1}^{10} 100(2^3) = 800$

$S_{10} = \dfrac{200(1-2^{10})}{1-2} = 204600$

10 c) $\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \dots$

$a = \frac{5}{6}$

$r = \frac{1}{3}$

$\left\{ \begin{array}{l} \left(\frac{5}{6} \times \frac{1}{3} \right) = \frac{5}{18} \\ \frac{5}{18} \times \frac{1}{3} = \frac{5}{54} \end{array} \right.$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{5}{6}}{1 - \frac{1}{3}} = \underline{\underline{\frac{5}{4}}}$$

d) $-1 < r < 1$

Leave blank

2. The fourth term of a geometric series is 10 and the seventh term of the series is 80.

For this series, find

(a) the common ratio, $\rightarrow r$

(2)

(b) the first term, $\rightarrow a$

(2)

(c) the sum of the first 20 terms, giving your answer to the nearest whole number.

(2)

Geometric progression

$a, ar, ar^2, ar^3, \ldots$

n^{th} term $= ar^{n-1}$

a) Given 4^{th} term $= 10$

$\therefore ar^3 = 10$ ①

Given 7^{th} term $= 80$

$\therefore ar^6 = 80$ ②

$\frac{②}{①}$ gives $\frac{ar^6}{ar^3} = \frac{80}{10}$

$\therefore r^3 = 8$

$\therefore r = \sqrt[3]{8}$

$\therefore r = 2$

b) Sub $r=2$ into ①

$a(2)^3 = 10$

$\therefore 8a = 10$

$a = \frac{10}{8} = \frac{5}{4} = 1.25$

c) $a = 1.25$, $r = 2$

$S_{20} = \frac{1.25(2^{20}-1)}{2-1}$

$= 1310718.75$

$= 1310719$

(to nearest whole number)

$S_n = \frac{a(r^n-1)}{r-1}$

or

$S_n = \frac{a(1-r^n)}{1-r}$

Leave blank

6. A car was purchased for £18 000 on 1st January.
On 1st January each following year, the value of the car is 80% of its value on 1st January in the previous year.

(a) Show that the value of the car exactly 3 years after it was purchased is £9216.
(1)

The value of the car falls below £1000 for the first time *n* years after it was purchased.

(b) Find the value of *n*.
(3)

An insurance company has a scheme to cover the maintenance of the car.
The cost is £200 for the first year, and for every following year the cost increases by 12% so that for the 3rd year the cost of the scheme is £250.88

(c) Find the cost of the scheme for the 5th year, giving your answer to the nearest penny.
(2)

(d) Find the total cost of the insurance scheme for the first 15 years.
(3)

a) $a = £18000$
$r = 0.8$

Price 3 years after purchase is ar^3
$= 18000 \times (0.8)^3 = £9216$

b) $18000 \times (0.8)^n < 1000$

$0.8^n < \frac{1000}{18000}$

$\log 0.8^n < \log \frac{1000}{18000}$

$n \log 0.8 < \log \frac{1000}{18000}$

$n > \frac{\log\left(\frac{1000}{18000}\right)}{\log 0.8}$

swap < to > as dividing by log 0.8 which is negative

$n > 12.95297$

$\therefore n = 13$

6c) Cost in 5^{th} year $= ar^4$

$a = 200, \quad r = 1.12$

$ar^4 = 200 \times (1.12)^4$

$= 314.70387$

$= £314.70$ to nearest penny

d) $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{15} = \dfrac{200(1-1.12^{15})}{1-1.12}$

$= 7455.9429$

$= £7455.94$ to nearest penny

C2 JAN 2011

Leave blank

3. The second and fifth terms of a geometric series are 750 and –6 respectively.

Find

(a) the common ratio of the series, **(3)**

(b) the first term of the series, **(2)**

(c) the sum to infinity of the series. **(2)**

1st 2nd 3rd

a) $a \quad ar \quad ar^2$

2nd $ar = 750$ ①

5th $ar^4 = -6$ ②

② ÷ ① gives $r^3 = -\frac{6}{750}$

$r = \sqrt[3]{-\frac{6}{750}} = -\frac{1}{5}$

b) $a = \frac{750}{-\frac{1}{5}} = -3750$

c) $S_\infty = \frac{a}{1-r} = \frac{-3750}{1--\frac{1}{5}}$

$= -3125$

H 3 5 4 0 3 A 0 6 2 8

C2 Jan 2012

Leave blank

1. A geometric series has first term $a = 360$ and common ratio $r = \frac{7}{8}$

Giving your answers to 3 significant figures where appropriate, find

(a) the 20th term of the series, **(2)**

(b) the sum of the first 20 terms of the series, **(2)**

(c) the sum to infinity of the series. **(2)**

a) $a = 360 \qquad r = \frac{7}{8}$

$ar^{19} = 360 \times \left(\frac{7}{8}\right)^{19}$

$= 28.474461$

$= 28.5 \ (3\,sf)$

b) $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{20} = \dfrac{360\left(1-\left(\frac{7}{8}\right)^{20}\right)}{1-\frac{7}{8}}$

$= 2680.6788$

$= 2680 \ (3\,sf)$

c) $S_\infty = \dfrac{a}{1-r} = \dfrac{360}{1-\frac{7}{8}} = 2880$

Leave blank

9. A geometric series has first term a and common ratio r.
The second term of the series is 4 and the sum to infinity of the series is 25.

(a) Show that $25r^2 - 25r + 4 = 0$. **(4)**

(b) Find the two possible values of r. **(2)**

(c) Find the corresponding two possible values of a. **(2)**

(d) Show that the sum, S_n, of the first n terms of the series is given by

$$S_n = 25(1 - r^n).$$

(1)

Given that r takes the larger of its two possible values,

(e) find the smallest value of n for which S_n exceeds 24. **(2)**

a) 1st term = a, common ratio = r,
2nd term = 4, sum to infinity = 25

Given $ar = 4$ ①
also $S_\infty = 25$
$\therefore \dfrac{a}{1-r} = 25$ ②

From ② $a = 25(1-r)$ ③

Sub ③ into ①
$25r(1-r) = 4$
$\therefore 25r - 25r^2 = 4$
$\therefore -25r + 25r^2 = -4$
$\therefore 25r^2 - 25r + 4 = 0$ as required

b) $\therefore (5r-1)(5r-4) = 0$
Either $5r - 1 = 0$ or $5r - 4 = 0$
$r = \frac{1}{5}$ $r = \frac{4}{5}$

9c) Sub $r = \frac{1}{5}$ into ③

$$a = 25\left(1 - \frac{1}{5}\right)$$

$$\therefore a = 20$$

Sub $r = \frac{4}{5}$ into ③

$$a = 25\left(1 - \frac{4}{5}\right)$$

$$a = 5$$

9d) $$S_n = \frac{a(1-r^n)}{1-r}$$

when $a = 20$, $r = \frac{1}{5}$

$$S_n = \frac{20(1 - r^n)}{1-\frac{1}{5}}$$

$$\therefore S_n = 25(1-r^n)$$

or

when $a = 5$, $r = \frac{4}{5}$

$$S_n = \frac{5(1-r^n)}{1-\frac{4}{5}}$$

$$\therefore S_n = 25(1-r^n)$$

e) S_n exceeds 24

$$\therefore S_n > 24$$

r takes larger value $\left(\frac{4}{5}\right)$

$$\therefore 25\left(1-\left(\frac{4}{5}\right)^n\right) > 24$$

$$1 - \left(\frac{4}{5}\right)^n > \frac{24}{25}$$

$$1 - \left(\frac{4}{5}\right)^n > 0.96$$

$$-\left(\frac{4}{5}\right)^n > -0.04$$

$$\left(\frac{4}{5}\right)^n < 0.04$$

× both sides by −1 reverses the inequality

$$\log 0.8^n < \log 0.04$$

$$n \log 0.8 < \log 0.04$$

$$n > \frac{\log 0.04}{\log 0.8}$$

dividing by −ve number inequality reverses

$$n > 14.25135$$

$\therefore$ smallest value of $n = 15$

8. A trading company made a profit of £50 000 in 2006 (Year 1).

A model for future trading predicts that profits will increase year by year in a geometric sequence with common ratio r, $r > 1$.

The model therefore predicts that in 2007 (Year 2) a profit of £50 000r will be made.

(a) Write down an expression for the predicted profit in Year n.

(1)

The model predicts that in Year n, the profit made will exceed £200 000.

(b) Show that $n > \dfrac{\log 4}{\log r} + 1$.

(3)

Using the model with $r = 1.09$,

(c) find the year in which the profit made will first exceed £200 000,

(2)

(d) find the total of the profits that will be made by the company over the 10 years from 2006 to 2015 inclusive, giving your answer to the nearest £10 000.

(3)

$a = 50000$ (Year 1)
$ar = 50000r$ (Year 2)
$ar^2 = 50000r^2$ (Year 3)
$ar^{n-1} = 50000r^{n-1}$ (Year n)

a) $50000r^{n-1}$

b) Year n profit exceeds £200000

$50000r^{n-1} > 200000$
$\log 50000 + \log r^{n-1} > \log 200000$
$\log r^{n-1} > \log 200000 - \log 50000$
$(n-1)\log r > \log \frac{200000}{50000}$
$(n-1)\log r > \log 4$
$n-1 > \frac{\log 4}{\log r}$
$n > \frac{\log 4}{\log r} + 1$ as required

Leave blank

8c) with $r = 1.09$

$$n > \frac{\log 4}{\log 1.09} + 1$$

$n > 17.086463$

As the year must be a whole number

Year 18

d) $$S_{10} = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{50000(1-1.09^{10})}{1-1.09}$$

$$= 759646.49$$

To the nearest 10000

£760000

Leave blank

6. A geometric series has first term 5 and common ratio $\frac{4}{5}$.

Calculate

(a) the 20th term of the series, to 3 decimal places, **(2)**

(b) the sum to infinity of the series. **(2)**

Given that the sum to k terms of the series is greater than 24.95,

(c) show that $k > \frac{\log 0.002}{\log 0.8}$, **(4)**

(d) find the smallest possible value of k. **(1)**

a) $a = 5, \ r = \frac{4}{5}$ $a, ar, ar^2, ar^3 \ldots$

20^{th} term $= ar^{19}$ n^{th} term $= ar^{n-1}$

$= 5 \times \left(\frac{4}{5}\right)^{19}$

$= 0.0720575$

$= 0.072$ (3dp)

b) $S_\infty = \frac{a}{1-r}, \ -1 < r < 1$

$S_\infty = \frac{5}{1-\frac{4}{5}} = \frac{5}{\frac{1}{5}} = 25$

c) Given $S_k > 24.95$ $S_n = \frac{a(1-r^n)}{1-r}$

$\therefore \frac{\overset{25}{\cancel{5}}\left(1-\left(\frac{4}{5}\right)^k\right)}{\cancel{1-\frac{4}{5}}_{\,1}} > 24.95$

$\therefore 25(1-0.8^k) > 24.95$

$\therefore 1 - 0.8^k > 0.998$

$\therefore 0.002 > 0.8^k$

$\therefore 0.8^k < 0.002$

$\therefore \log 0.8^k < \log 0.002$

$\therefore k \log 0.8 < \log 0.002$

$\therefore k > \frac{\log 0.002}{\log 0.8}$

When you divide any inequality by a NEGATIVE number, you MUST REVERSE THE SIGN

log of any number less than 1 is $-$ve

6d) $k > \dfrac{\log 0.002}{\log 0.8}$

$\therefore k > 27.8502\ldots$

k is an integer though

$\therefore$ smallest value of $k = 28$

Leave blank

5. The third term of a geometric sequence is 324 and the sixth term is 96

(a) Show that the common ratio of the sequence is $\frac{2}{3}$ **(2)**

(b) Find the first term of the sequence. **(2)**

(c) Find the sum of the first 15 terms of the sequence. **(3)**

(d) Find the sum to infinity of the sequence. **(2)**

Third term is $ar^2 = 324$ ①
Sixth term is $ar^5 = 96$ ②

a) ② ÷ ① gives
$$r^3 = \frac{96}{324}$$
$$r = \sqrt[3]{\frac{96}{324}} = \frac{2}{3}$$

b) Using ① above

First term is a

$$ar^2 = 324$$
$$a \times \left(\frac{2}{3}\right)^2 = 324$$
$$a = \frac{324 \times 9}{4}$$
$$a = 729$$

c) $$S_{15} = \frac{a(1-r^n)}{1-r} = \frac{729(1-\frac{2}{3}^{15})}{1-\frac{2}{3}}$$

$$S_{15} = 2182.0056$$

d) $$S_\infty = \frac{a}{1-r} = \frac{729}{1-\frac{2}{3}} = 2187$$

9a) continued

② gives $h = \frac{600}{r^2}$

sub h in ① gives

$$S = r^2 + 3\cancel{r}^{1} \times \frac{600}{r^{\cancel{2}}}$$

$$S = r^2 + \frac{1800}{r}$$ as required

$S = r^2 + 1800r^{-1}$

b) $\frac{dS}{dr} = 2r - 1800r^{-2}$

$$\frac{dS}{dr} = 2r - \frac{1800}{r^2}$$

For S stationary

$$\frac{dS}{dr} = 0$$

$$0 = 2r - \frac{1800}{r^2}$$

$$\frac{1800}{r^2} = 2r$$

$$2r^3 = 1800$$

$$r^3 = 900$$

$$r = \sqrt[3]{900} = 9.654$$

$r = 9.7\text{cm}$ (1dp)

c) $\frac{d^2S}{dr^2} = 2 + 3600r^{-3}$

$$\frac{d^2S}{dr^2} = 2 + \frac{3600}{r^3}$$

when $r = 9.7\text{ cm}$, $\frac{d^2S}{dr^2} = 10$

$\frac{d^2S}{dr^2}$ is positive so this gives a minimum value for S

9.

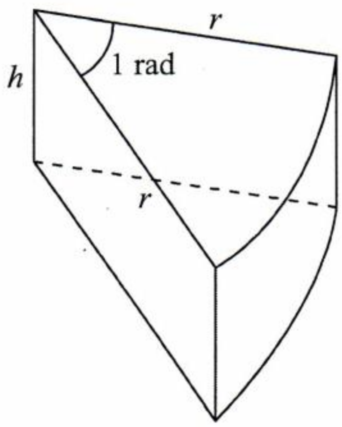

Figure 2

Figure 2 shows a closed box used by a shop for packing pieces of cake. The box is a right prism of height h cm. The cross section is a sector of a circle. The sector has radius r cm and angle 1 radian.

The volume of the box is 300 cm^3.

(a) Show that the surface area of the box, S cm^2, is given by

$$S = r^2 + \frac{1800}{r}$$

(5)

(b) Use calculus to find the value of r for which S is stationary.

(4)

(c) Prove that this value of r gives a minimum value of S.

(2)

(d) Find, to the nearest cm^2, this minimum value of S.

(2)

a) Arc length $= r\theta = r \times 1 = r$

Sector area $= \frac{1}{2}r^2\theta = \frac{1}{2} \times r^2 \times 1 = \frac{1}{2}r^2$

Area rectangle $= hr$ (2 of these)

Area of curved face (rectangle) $= r \times h$

Surface area $=$ 2 sectors + 2 rectangles + curved surface

$= 2 \times (\frac{1}{2}r^2) + 2rh + rh$

Surface area $= r^2 + 3rh$ (1)

Volume $= 300 =$ Area of cross-section $\times$ height

Volume $= \frac{1}{2}r^2 h = 300$ (2)

9d) $S = r^2 + \frac{1800}{r}$

$S = (9.6548938)^2 + \frac{1800}{9.6548938}$

$S = 279.65093$

$S = 280 \text{ cm}^2$ (nearest cm^2)

C2 June 2010

Leave blank

9. The adult population of a town is 25 000 at the end of Year 1.

A model predicts that the adult population of the town will increase by 3% each year, forming a geometric sequence.

(a) Show that the predicted adult population at the end of Year 2 is 25 750.
(1)

(b) Write down the common ratio of the geometric sequence.
(1)

The model predicts that Year N will be the first year in which the adult population of the town exceeds 40 000.

(c) Show that

$$(N-1)\log 1.03 > \log 1.6$$

(3)

(d) Find the value of N.
(2)

At the end of each year, each member of the adult population of the town will give £1 to a charity fund.

Assuming the population model,

(e) find the total amount that will be given to the charity fund for the 10 years from the end of Year 1 to the end of Year 10, giving your answer to the nearest £1000.
(3)

a) $a = 25000$
$r = 1.03$ (increase of 3%)

$ar = 25000 \times 1.03 = 25750$ as required

b) $r = 1.03$

c) Year1 Year2 Year 3 Year N
a ar ar^2 ar^{N-1}

$ar^{N-1} > 40000$
$25000 \times 1.03^{N-1} > 40000$
$1.03^{N-1} > \frac{40000}{25000}$

9c) continued

$1.03^{N-1} > 1.6$

$\log 1.03^{N-1} > \log 1.6$

$(N-1)\log 1.03 > \log 1.6$ as required

d) $N-1 > \dfrac{\log 1.6}{\log 1.03}$

$N > 1 + \dfrac{\log 1.6}{\log 1.03}$

$N > 16.900632$

$\therefore N = 17$ as must be integer

e) $S_{10} = \dfrac{a(1-r^{10})}{1-r}$

$= \dfrac{25000\,(1-1.03^{10})}{1-1.03}$

$= 286596.98$

$= £287000$ (to nearest £1000)

May 2011

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(a) the common ratio, **(2)**

(b) the first term, **(2)**

(c) the sum to infinity, **(2)**

(d) the smallest value of n for which the sum of the first n terms of the series exceeds 1000. **(4)**

Leave blank

$a \qquad ar \qquad ar^2$

$\qquad 192 \qquad 144$

a) $\frac{ar^2}{ar} = r = \frac{144}{192} = \frac{3}{4}$

b) $a = 192 \div \frac{3}{4} = 256$

c) $S_\infty = \frac{a}{1-r} = \frac{256}{1-\frac{3}{4}} = 1024$

d) $S_n = \frac{a(1-r^n)}{1-r}$

we want $\frac{a(1-r^n)}{1-r} > 1000$

$\frac{256(1-0.75^n)}{1-0.75} > 1000$

$256(1-0.75^n) > 1000 \times 0.25$

P38158A01632

May 2011

Leave blank

Question 6 continued

$1 - 0.75^n > \frac{250}{256}$

$1 - \frac{250}{256} > 0.75^n$

$\log\left(1 - \frac{250}{256}\right) > \log 0.75^n$

$\log\left(1 - \frac{250}{256}\right) > n \log 0.75$

$\frac{\log\left(1 - \frac{250}{256}\right)}{\log 0.75} < n$

swap inequality sign as $\log 0.75$ is $-$ve

$n > 13.047$

So smallest value of n is 14

Leave blank

9. A geometric series is $a + ar + ar^2 + \ldots$

(a) Prove that the sum of the first n terms of this series is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

The third and fifth terms of a geometric series are 5.4 and 1.944 respectively and all the terms in the series are positive.

For this series find,

(b) the common ratio,

(2)

(c) the first term,

(2)

(d) the sum to infinity.

(3)

a) $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ ①

$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$ ②

① − ② gives

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$S_n = \dfrac{a(1-r^n)}{1-r}$ as required

b) $ar^2 = 5.4$

$ar^4 = 1.944$

$\dfrac{ar^4}{ar^2} = \dfrac{1.944}{5.4}$

$r^2 = 0.36$

$r = 0.6$

c) $ar^2 = 5.4$

$a \times 0.6^2 = 5.4$

$a = \dfrac{5.4}{0.6^2} = 15$

P 4 0 6 8 5 A 0 2 6 2 8

Leave blank

9. A geometric series is $a + ar + ar^2 + \ldots$

(a) Prove that the sum of the first n terms of this series is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$ **(4)**

The third and fifth terms of a geometric series are 5.4 and 1.944 respectively and all the terms in the series are positive.

For this series find,

(b) the common ratio, **(2)**

(c) the first term, **(2)**

(d) the sum to infinity. **(3)**

a) $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ ①
$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$ ②

① − ② gives

$S_n - rS_n = a - ar^n$
$S_n(1-r) = a(1-r^n)$
$S_n = \frac{a(1-r^n)}{1-r}$ as required

b) $ar^2 = 5.4$
$ar^4 = 1.944$

$\frac{ar^4}{ar^2} = \frac{1.944}{5.4}$

$r^2 = 0.36$
$r = 0.6$

c) $ar^2 = 5.4$
$a \times 0.6^2 = 5.4$
$a = \frac{5.4}{0.6^2} = 15$

P 4 0 6 8 5 A 0 2 6 2 8

C2 Jan 2013

Leave blank

3. A company predicts a yearly profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%. The predicted yearly profit forms a geometric sequence with common ratio 1.05

(a) Show that the predicted profit in the year 2016 is £138 915

(1)

(b) Find the first year in which the yearly predicted profit exceeds £200 000

(5)

(c) Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound.

(3)

a) $a = 120000$ 1st 2nd 3rd 4th

$r = 1.05$ 2013 2014 2015 2016

Year 2016 is 4th term

$ar^3 = 120000 \times 1.05^3 = £138915$

as required

b) $ar^{n-1} > 200000$

$120000 \times 1.05^{n-1} > 200000$

$1.05^n > \frac{200000}{120000}$

$\log 1.05^{n-1} > \log \frac{200000}{120000}$

$(n-1)\log 1.05 > \log \frac{200000}{120000}$

$n-1 > \frac{\log \frac{200000}{120000}}{\log 1.05}$

$n > 10.469848 + 1$

$n > 11.469848$

must be integer so $n = 12$

1st → 2013 2nd → 2014 → 12th → 2024

Year when exceeds £200000 is 2024

C2 Jan 2013

3c) 2013 → 2023 is year 1 to Year 11 total

$$S_n = \frac{a(1-r^n)}{1-r} \qquad n=11$$

$$S_{11} = \frac{120000(1-1.05^{11})}{1-1.05}$$

$$S_{11} = 1704814.5$$

$$= \underline{\underline{£1704815}} \quad \text{(to nearest pound)}$$

CZ MAY 2013

Leave blank

1. The first three terms of a geometric series are

18, 12 and p

respectively, where p is a constant.

Find

(a) the value of the common ratio of the series,
(1)

(b) the value of p,
(1)

(c) the sum of the first 15 terms of the series, giving your answer to 3 decimal places.
(2)

1st 2nd 3rd

18 12 p

a ar ar^2

a) $r = \frac{ar}{a} = \frac{12}{18} = \underline{\underline{\frac{2}{3}}}$

b) $\frac{12}{18} = \frac{p}{12}$

$p = \frac{12 \times 12}{18} = \underline{\underline{8}}$

c) $S_{15} = \frac{a(1-r^n)}{1-r} = \frac{18\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}}$

$= 53.876688$

$= \underline{\underline{53.877 \text{ (3dp)}}}$

P 4 1 8 5 9 A 0 2 3 2