

5.

$$f(x) = x^3 + 4x^2 + x - 6.$$

- (a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$.

(2)

- (b) Factorise $f(x)$ completely.

(4)

- (c) Write down all the solutions to the equation

$$x^3 + 4x^2 + x - 6 = 0.$$

(1)

(a) for $(x+2)$

$$\begin{aligned} f(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 \\ &= -8 + 16 - 2 - 6 \\ &= 0 \quad \text{so } (x+2) \text{ is a factor} \end{aligned}$$

5)

$$\begin{array}{r} x+2 \mid \begin{array}{r} x^3 + 2x^2 - 3 \\ x^3 + 4x^2 + x - 6 \\ \hline -x^3 - 2x^2 \\ \hline 2x^2 + x \\ \hline -2x^2 - 4x \\ \hline -3x - 6 \\ \hline 0 \end{array} \end{array}$$

$$\therefore f(x) = (x+2)(x^2 + 2x - 3)$$

$$f(x) = (x+2)(x+3)(x-1)$$

c) Either $x+2=0$ or $x+3=0$ or $x-1=0$

$$x = -2 \qquad \qquad x = -3 \qquad \qquad x = 1$$



1. (a) Find the remainder when

$$x^3 - 2x^2 - 4x + 8$$

is divided by

(i) $x - 3$,

(ii) $x + 2$.

Remainder theorem

- if $f(x)$

divided by
 $x - a$

then
remainder = $f(a)$

(3)

or if
 $f(x)$ divided
by $x + a$

remainder
= $f(-a)$

(4)

- (b) Hence, or otherwise, find all the solutions to the equation

$$x^3 - 2x^2 - 4x + 8 = 0.$$

a) let $f(x) = x^3 - 2x^2 - 4x + 8$

(i) remainder = $f(3)$ for $x = 3$

$$\begin{aligned} &= (3)^3 - 2(3)^2 - 4(3) + 8 \\ &= 5 \end{aligned}$$

(ii) remainder = $f(-2)$
 $= (-2)^3 - 2(-2)^2 - 4(-2) + 8$
 $= 0$

b) As $f(-2) = 0$ then $x + 2$ is a factor

Quadratic factor $x^2 - 4x + 4$

$$f(x) = (x+2)(\quad)$$

$$\begin{array}{r} x+2 \quad | \quad x^3 - 2x^2 - 4x + 8 \\ \underline{-} \quad x^3 + 2x^2 \quad \downarrow \\ \underline{-} \quad -4x^2 - 4x \quad | \\ \underline{-} \quad -4x^2 - 8x \quad \downarrow \\ \underline{\quad \quad \quad 4x + 8} \\ \underline{\quad \quad \quad 4x + 8} \\ 0 \end{array}$$

$$\therefore (?) = \frac{f(x)}{x+2}$$

$$\therefore f(x) = (x+2)(x^2 - 4x + 4)$$

$$f(x) = (x+2)(x-2)(x-2)$$

$$\text{When } f(x) = 0 \quad \therefore x+2 = 0 \text{ or } x-2 = 0$$

$$\therefore x = -2 \text{ or } x = 2$$

6.

$$f(x) = x^4 + 5x^3 + ax + b,$$

where a and b are constants.

The remainder when $f(x)$ is divided by $(x - 2)$ is equal to the remainder when $f(x)$ is divided by $(x + 1)$.

(a) Find the value of a .

(5)

Given that $(x + 3)$ is a factor of $f(x)$,

(b) find the value of b .

(3)

a) let $f(x) = x^4 + 5x^3 + ax + b$

Remainder for when $f(x)$ divided by $(x - 2)$
 $f(2) = 2^4 + 5(2^3) + 2a + b$
 $= 16 + 40 + 2a + b$
 $= 56 + 2a + b \quad (1)$

Remainder when $f(x)$ divided by $(x + 1)$
 $f(-1) = (-1)^4 + 5(-1)^3 - a + b$
 $= 1 - 5 - a + b$
 $= -4 - a + b \quad (2)$

As remainders are equal, solve (1) and (2) as simultaneous equations

$$\begin{aligned} 56 + 2a + b &= -4 - a + b \\ 3a &= -60 \\ a &= -20 \end{aligned}$$



6 b) $(x+3)$ is a factor for $f(-3)=0$

$$0 = (-3)^4 + 5(-3)^3 - 20(-3) + b$$

$$0 = 81 - 135 + 60 + b$$

$$b = -81 + 135 - 6$$

$$b = -6$$

3.

$$f(x) = 2x^3 + ax^2 + bx - 6$$

where a and b are constants.

When $f(x)$ is divided by $(2x - 1)$ the remainder is -5 .

When $f(x)$ is divided by $(x + 2)$ there is no remainder.

(a) Find the value of a and the value of b .

(6)

(b) Factorise $f(x)$ completely.

(3)

a)

When $f(x)$ divided by $2x - 1$ remainder -5

$$\begin{aligned} -5 &= f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) - 6 \\ \therefore -5 &= \frac{1}{4} + \frac{1}{4}a + \frac{1}{2}b - 6 \\ \frac{3}{4} &= \frac{1}{4}a + \frac{1}{2}b \quad (1) \end{aligned}$$

When $f(x)$ divided by $x + 2$ remainder 0

$$\begin{aligned} 0 &= f(-2) = 2(-2)^3 + a(-2)^2 + b(-2) - 6 \\ 0 &= -16 + 4a - 2b - 6 \\ 22 &= 4a - 2b \quad (2) \end{aligned}$$

Solve (1) and (2) simultaneously

$$\begin{array}{rcl} (1) \times 4 & 3 = a + 2b & (3) \\ & 22 = 4a - 2b & (2) \\ (3) + (2) & 25 = 5a & \\ & a = 5 & \end{array}$$

$$\begin{array}{l} \text{put } a = 5 \text{ in (3)} \\ 3 = 5 + 2b \\ -2 = 2b \\ b = -1 \end{array}$$

$$\begin{array}{l} \text{Check in (2)} \\ 4 \times 5 - 2(-1) \\ = 22 \checkmark \end{array}$$

$$\therefore a = 5, b = -1$$

$$3b) f(x) = 2x^3 + 5x^2 - x - 6$$

we know $(x+2)$ is a factor

$$\begin{array}{r} 2x^2 + x - 3 \\ \hline x + 2 | 2x^3 + 5x^2 - x - 6 \\ \quad - \underline{2x^3 + 4x^2} \downarrow \\ \quad \quad \quad x^2 - x \\ \quad \quad \quad - \underline{x^2 + 2x} \downarrow \\ \quad \quad \quad - 3x - 6 \\ \quad \quad \quad - 3x - 6 \\ \hline 0 \end{array}$$

$$f(x) = (x+2)(2x^2 + x - 3)$$

$$f(x) = (x+2)(2x+3)(x-1)$$

factorised completely

1. $f(x) = x^4 + x^3 + 2x^2 + ax + b$

where a and b are constants.

When $f(x)$ is divided by $(x - 1)$, the remainder is 7.

(a) Show that $a + b = 3$.

(2)

When $f(x)$ is divided by $(x + 2)$, the remainder is -8.

(b) Find the value of a and the value of b .

(5)

$$\begin{aligned} \text{a)} \quad f(1) &= 1^4 + 1^3 + 2 \times 1^2 + a + b = 7 \\ 7 &= 1 + 1 + 2 + a + b \\ 7 - 1 - 1 - 2 &= a + b \\ 3 &= a + b \quad \text{as required} \end{aligned}$$

①

$$\text{b)} \quad f(-2) = -8$$

$$\begin{aligned} (-2)^4 + (-2)^3 + 2(-2)^2 - 2a + b &= -8 \\ 16 - 8 + 8 - 2a + b &= -8 \\ -2a + b &= -24 \end{aligned}$$

②

Solve ① and ② simultaneously

$$\begin{aligned} \text{① gives } a &= 3 - b \\ \text{sub in ② } -2(3 - b) + b &= -24 \\ -6 + 2b + b &= -24 \\ 3b &= -18 \\ b &= -6 \end{aligned}$$

$$\text{in ① gives } a = 9$$

$$a = 9, b = -6$$



5. $f(x) = x^3 + ax^2 + bx + 3$, where a and b are constants.

Given that when $f(x)$ is divided by $(x+2)$ the remainder is 7,

(a) show that $2a - b = 6$

(2)

Given also that when $f(x)$ is divided by $(x-1)$ the remainder is 4,

(b) find the value of a and the value of b .

(4)

a) $f(-2) = 7$
 $(-2)^3 + a(-2)^2 + b(-2) + 3 = 7$
 $-8 + 4a - 2b + 3 = 7$
 $4a - 2b - 5 = 7$
 $4a - 2b = 12$
 $(\div \text{ through by } 2)$ $2a - b = 6$ (as required) ①

b) $f(1) = 4$
 $1 + a + b + 3 = 4$
 $a + b + 4 = 4$
 $a + b = 0$
 $a = -b$

in ① $-2b - b = 6$
 $-3b = 6$
 $b = -2$
 $a = 2$



4.

$$f(x) = 2x^3 + 3x^2 - 29x - 60.$$

(a) Find the remainder when $f(x)$ is divided by $(x + 2)$.

(2)

(b) Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

(2)

(c) Factorise $f(x)$ completely.

$$\begin{aligned} \text{a)} \quad f(-2) &= 2(-2)^3 + 3(-2)^2 - 29(-2) - 60 \\ &= -16 + 12 + 58 - 60 \\ &= -6 \end{aligned} \quad (4)$$

$$\begin{aligned} \text{b)} \quad f(-3) &= 2(-3)^3 + 3(-3)^2 - 29(-3) - 60 \\ &= -54 + 27 + 87 - 60 \\ &= 0 \end{aligned}$$

$\therefore x+3$ is a factor of $f(x)$
as remainder is zero

$$\begin{array}{r} 2x^2 - 3x - 20 \\ \hline x + 3 | 2x^3 + 3x^2 - 29x - 60 \\ \quad - 2x^3 - 6x^2 \quad \downarrow \quad \downarrow \\ \hline \quad \quad - 3x^2 - 29x \\ \quad - \quad - 3x^2 - 9x \\ \hline \quad \quad \quad - 20x - 60 \\ \quad - \quad - 20x - 60 \\ \hline \quad \quad \quad \quad 0 \end{array}$$

$$\begin{aligned} \therefore f(x) &\equiv (x+3)(2x^2 - 3x - 20) \\ &\equiv (x+3)(2x+5)(x-4) \end{aligned}$$



2.

$$f(x) = 3x^3 - 5x^2 - 16x + 12.$$

- (a) Find the remainder when $f(x)$ is divided by $(x - 2)$.

(2)

Given that $(x + 2)$ is a factor of $f(x)$,

- (b) factorise $f(x)$ completely.

(4)

a) for $(x - 2)$

$$\begin{aligned} f(2) &= 3(2)^3 - 5(2)^2 - 16(2) + 12 \\ &= 24 - 20 - 32 + 12 \\ &= -16 \end{aligned}$$

Remainder is -16

b)

$$\begin{array}{r} 3x^2 - 11x + 6 \\ \hline x + 2 | 3x^3 - 5x^2 - 16x + 12 \\ - \underline{3x^3 + 6x^2} \\ \hline -11x^2 - 16x \\ + \underline{11x^2 + 22x} \\ \hline 6x + 12 \\ \hline 6x + 12 \\ \hline 0 \end{array}$$

$$\begin{aligned} \therefore f(x) &= (x+2)(3x^2 - 11x + 6) \\ &= (x+2)(3x-2)(x-3) \end{aligned}$$

Q2

(Total 6 marks)

1.

$$f(x) = 2x^3 - 3x^2 - 39x + 20$$

- (a) Use the factor theorem to show that $(x + 4)$ is a factor of $f(x)$.

(2)

- (b) Factorise $f(x)$ completely.

(4)

a) if $(x+4)$ is a factor

$$\begin{aligned} f(-4) &= 2(-4)^3 - 3(-4)^2 \\ &\quad - 39(-4) + 20 \\ &= -128 - 48 + 156 + 20 \\ &= 0 \end{aligned}$$

$\therefore x+4$ is a factor of $f(x)$

b) $f(x) \equiv 2x^3 - 3x^2 - 39x + 20$

For the quadratic factor

divide by $x+4$
using long division

Factor theorem

$$\begin{aligned} f(-a) &= 0 \\ \therefore (x+a) &\text{ is} \\ &\text{a factor} \\ &\text{of } f(x) \end{aligned}$$

$$\begin{aligned} f(x) &= (x+4)(x^2 + \dots) \\ (?) &\equiv \frac{f(x)}{x+4} \end{aligned}$$

$$\begin{array}{r} 2x^2 - 11x + 5 \\ \hline x+4 | 2x^3 - 3x^2 - 39x + 20 \\ \quad - 2x^3 + 8x^2 \\ \hline \quad \quad - 11x^2 - 39x \\ \quad - \quad - 11x^2 - 44x \\ \hline \quad \quad \quad 5x + 20 \\ \quad \quad - 5x - 20 \\ \hline \quad \quad \quad 0 \end{array}$$

$$\begin{aligned} \therefore f(x) &\equiv (x+4)(2x^2 - 11x + 5) \\ &\equiv (x+4)(2x-1)(x-5) \end{aligned}$$



3.

$$f(x) = (3x - 2)(x - k) - 8$$

where k is a constant.

- (a) Write down the value of $f(k)$.

(1)

When $f(x)$ is divided by $(x - 2)$ the remainder is 4

- (b) Find the value of k .

(2)

- (c) Factorise $f(x)$ completely.

(3)

$$\begin{aligned} a) \quad f(k) &= (3k - 2)(k - k) - 8 \\ &= (3k - 2)(0) - 8 \\ &= -8 \end{aligned}$$

$$\begin{aligned} b) \quad f(2) &= 4 \\ f(2) &= (3 \times 2 - 2)(2 - k) - 8 = 4 \\ \therefore (4)(2 - k) - 8 &= 4 \\ 8 - 4k - 8 &= 4 \\ -4k &= 4 \\ k &= -1 \end{aligned}$$

$$\begin{aligned} c) \quad f(x) &= (3x - 2)(x - -1) - 8 \\ f(x) &= (3x - 2)(x + 1) - 8 \\ f(x) &= 3x^2 + 3x - 2x - 2 - 8 \\ f(x) &= 3x^2 + x - 10 \\ f(x) &= (3x - 5)(x + 2) \end{aligned}$$



2.

$$f(x) = 3x^3 - 5x^2 - 58x + 40$$

- (a) Find the remainder when $f(x)$ is divided by $(x - 3)$.

(2)

Given that $(x - 5)$ is a factor of $f(x)$,

- (b) find all the solutions of $f(x) = 0$.

$$\text{a) } f(3) = 3(3)^3 - 5(3)^2 - 58(3) + 40 \quad (5)$$

$$= 81 - 45 - 174 + 40$$

$$= -98$$

b) $\frac{3x^2 + 10x - 8}{x - 5}$

$$\begin{array}{r}
 3x^2 + 10x - 8 \\
 \hline
 x - 5 \left| \begin{array}{r} 3x^3 - 5x^2 - 58x + 40 \\ - 3x^3 + 15x^2 \quad \downarrow \\ \hline 10x^2 - 58x \quad \downarrow \\ - 10x^2 + 50x \quad \hline - 8x + 40 \\ \hline 0 \end{array} \right.
 \end{array}$$

Factorise

$$(3x^2 + 10x - 8) = (3x-2)(x+4)$$

$$\therefore (3x - 2)(x + 4)(x - 5) = 0$$

$$x = \frac{2}{3}, x = -4, x = 5$$



1.

$$f(x) = 2x^3 - 7x^2 - 5x + 4$$

(a) Find the remainder when $f(x)$ is divided by $(x-1)$.

(2)

(b) Use the factor theorem to show that $(x+1)$ is a factor of $f(x)$.

(2)

(c) Factorise $f(x)$ completely.

(4)

$$\text{a) } f(1) = 2(1)^3 - 7(1)^2 - 5(1) + 4 \\ = 2 - 7 - 5 + 4 \\ = -6$$

$$\text{b) } f(-1) = 2(-1)^3 - 7(-1)^2 - 5(-1) + 4 \\ = -2 - 7 + 5 + 4 \\ = 0$$

as $f(-1) = 0$
then $x+1$ is a factor.

$$\text{c) } \begin{array}{r} 2x^2 - 9x + 4 \\ x+1 \overline{)2x^3 - 7x^2 - 5x + 4} \\ - \underline{2x^3 + 2x^2} \\ - 9x^2 - 5x \\ - \underline{- 9x^2 - 9x} \\ 4x + 4 \end{array}$$

$$2x^2 - 9x + 4 \\ (2x - 1)(x - 4)$$

$$\therefore f(x) = (x+1)(2x-1)(x-4)$$



4.

$$f(x) = 2x^3 - 7x^2 - 10x + 24$$

- (a) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$.

(2)

- (b) Factorise $f(x)$ completely.

(4)

$$\begin{aligned} a) \quad f(-2) &= 2(-2)^3 - 7(-2)^2 \\ &\quad - 10(-2) + 24 \\ &= -16 - 28 + 20 + 24 \\ &= 0 \end{aligned}$$

as $f(-2) = 0$ then $x+2$ is a factor of $f(x)$

b)

$$\begin{array}{r} 2x^2 - 11x + 12 \\ \hline x+2 | 2x^3 - 7x^2 - 10x + 24 \\ \quad - 2x^3 + 4x^2 \quad \downarrow \\ \quad \quad \quad - 11x^2 - 10x \quad \downarrow \\ \quad \quad \quad - -11x^2 - 22x \\ \quad \quad \quad \quad \quad 12x + 24 \\ \quad \quad \quad \quad - 12x + 24 \\ \hline \quad \quad \quad \quad \quad 0 \end{array}$$

Factorise $2x^2 - 11x + 12$

$$(2x - 3)(x - 4)$$

$$f(x) = (x+2)(2x-3)(x-4)$$



2. $f(x) = ax^3 + bx^2 - 4x - 3$, where a and b are constants.

Given that $(x - 1)$ is a factor of $f(x)$,

(a) show that

$$a + b = 7 \quad (2)$$

Given also that, when $f(x)$ is divided by $(x + 2)$, the remainder is 9,

(b) find the value of a and the value of b , showing each step in your working. (4)

a) $(x - 1)$ is a factor, so $f(1) = 0$

$$f(1) = 0 = ax^3 + bx^2 - 4x - 3$$

$$0 = a + b - 4 - 3$$

$$a + b = 7 \quad (1) \text{ as required}$$

b) $f(-2) = 9$

$$9 = (-2)^3 a + (-2)^2 b - 4(-2) - 3$$

$$9 = -8a + 4b + 8 - 3$$

$$9 - 8 + 3 = -8a + 4b$$

$$4 = -8a + 4b \quad (2)$$

using $a = 7 - b$ from (1) in (2)

$$4 = -8(7 - b) + 4b$$

$$4 = -56 + 8b + 4b$$

$$60 = 12b$$

$$b = 5$$

in (1) $a = 2$

$$\underline{\underline{a = 2}}, \underline{\underline{b = 5}}$$



3.

$$f(x) = 2x^3 - 5x^2 + ax + 18$$

where a is a constant.

Given that $(x - 3)$ is a factor of $f(x)$,

(a) show that $a = -9$

(2)

(b) factorise $f(x)$ completely.

(4)

Given that

$$g(y) = 2(3^{3y}) - 5(3^{2y}) - 9(3^y) + 18$$

(c) find the values of y that satisfy $g(y) = 0$, giving your answers to 2 decimal places where appropriate.

(3)

a) If $(x - 3)$ is a factor, $f(3) = 0$

$$0 = 2 \times 3^3 - 5 \times 3^2 + 3a + 18$$

$$0 = 54 - 45 + 3a + 18$$

$$-54 + 45 - 18 = 3a$$

$$-27 = 3a$$

$$\underline{\underline{a = -9}}$$

as required

b)

$$\begin{array}{r} 2x^2 + x - 6 \\ \hline x - 3 | 2x^3 - 5x^2 - 9x + 18 \\ \quad - \underline{2x^3 - 6x^2} \\ \quad \quad \quad \downarrow \\ \quad \quad \quad x^2 - 9x \\ \quad \quad \quad - \underline{x^2 - 3x} \\ \quad \quad \quad \downarrow \\ \quad \quad \quad -6x + 18 \\ \quad \quad \quad -6x + 18 \\ \hline \quad \quad \quad 0 \end{array}$$

$$2x^2 + x - 6 = (2x - 3)(x + 2)$$

$$f(x) = (x - 3)(x + 2)(2x - 3)$$

fully factorised



$$3c) g(y) = 2(3^{3y}) - 5(3^{2y}) - 9(3^y) + 18$$

$$\text{let } x = 3^y$$

and then it matches

$$f(x) = 2x^3 - 5x^2 - 9x + 18$$

$$f(x) = (x-3)(x+2)(2x-3)$$

$$0 = (x-3)(x+2)(2x-3)$$

$$\text{solutions are } x=3, x=-2, x=\frac{3}{2}$$

$$\text{but } 3^y = x$$

$$3^y = 3 \quad , \quad \log 3^y = \log 3 \\ y \log 3 = \log 3 \\ y = \frac{\log 3}{\log 3} = 1$$

$$3^y = -2$$

$$\log 3^y = \log(-2) \\ y \log 3 = \log(-2)$$

$$y = \frac{\log(-2)}{\log 3} \quad (\text{error})$$

-no
solution,
can't have
 $\log(-2)$)

$$3^y = 1.5$$

$$y \log 3 = \log 1.5$$

$$y = \frac{\log 1.5}{\log 3} = 0.3690702$$

Solutions are $y=1$ and $y=0.37$ (2dp)