**5.** The radioactive decay of a substance is given by

$$R = 1000e^{-ct}, \qquad t \geqslant 0.$$

where $R$ is the number of atoms at time $t$ years and $c$ is a positive constant.

(a) Find the number of atoms when the substance started to decay.

**(1)**

It takes 5730 years for half of the substance to decay.

(b) Find the value of $c$ to 3 significant figures.

**(4)**

(c) Calculate the number of atoms that will be left when $t = 22\ 920$ .

**(2)**

(d) In the space provided on page 13, sketch the graph of $R$ against $t$ .

**(2)**

a) $t = 0 \quad R = 1000$

b) $t = 5730 \quad R = 500$

$500 = 1000e^{-c.5730}$

$\frac{1}{2} = e^{-5730c}$

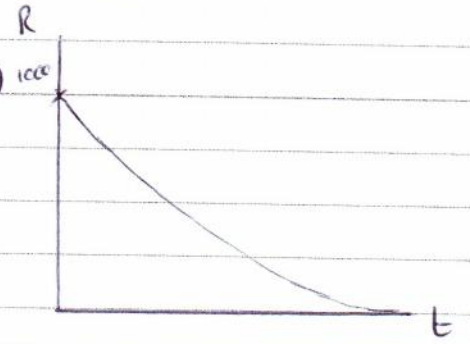
$\ln(1/2) = -5730c$

$c = 0.000121$

c) $R = 1000e^{-c.(22920)}$ do not round c from previous ans.

$R = 62.5$

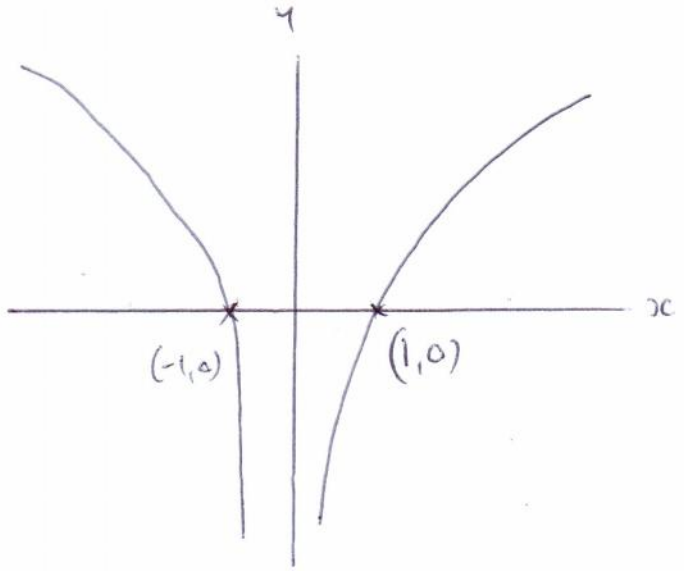
d)



Leave blank

**5.** Sketch the graph of $y = \ln|x|$, stating the coordinates of any points of intersection with the axes.

**(3)**



Leave blank

**9.** (i) Find the exact solutions to the equations

(a) $\ln(3x - 7) = 5$

**(3)**

(b) $3^x e^{7x+2} = 15$

**(5)**

(ii) The functions f and g are defined by

$$f(x) = e^{2x} + 3, \qquad x \in \mathbb{R}$$
$$g(x) = \ln(x - 1), \qquad x \in \mathbb{R},\ x > 1$$

(a) Find $f^{-1}$ and state its domain.

**(4)**

(b) Find fg and state its range.

**(3)**

9) (i) a) $e^{\ln(3x-7)} = e^5$

$3x - 7 = e^5$

$x = \frac{e^5 + 7}{3}$

(b) $3^x e^{7x+2} = 15$

$\ln(3^x e^{7x+2}) = \ln 15$

$\ln 3^x + \ln e^{7x+2} = \ln 15$

$x \ln 3 + 7x + 2 = \ln 15$

$x(\ln 3 + 7) = -2 + \ln 15$

$x = \frac{-2 + \ln 15}{\ln 3 + 7}$.

Leave blank

**Question 9 continued**

(ii) $f(x) = e^{2x} + 3$

$y = e^{2x} + 3$

$y - 3 = e^{2x}$

$\ln(y-3) = \ln e^{2x}$

$\ln(y-3) = 2x$

$\frac{\ln(y-3)}{2} = x$

$f^{-1}(x) = \frac{\ln(x-3)}{2}$

Domain $x > 3$

(b) $g(x) = \ln(x-1)$

$fg(x) = e^{2\ln(x-1)} + 3$

$= (x-1)^2 + 3$

Range $y > 3$

**4.** Joan brings a cup of hot tea into a room and places the cup on a table. At time $t$ minutes after Joan places the cup on the table, the temperature, $\theta$ °C, of the tea is modelled by the equation

$$\theta = 20 + Ae^{-kt},$$

where $A$ and $k$ are positive constants.

Given that the initial temperature of the tea was 90 °C,

(*a*) find the value of $A$.

**(2)**

The tea takes 5 minutes to decrease in temperature from 90 °C to 55 °C.

(*b*) Show that $k = \frac{1}{5}\ln 2$.

**(3)**

(*c*) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in °C per minute, to 3 decimal places.

**(3)**

---

4 a) $\theta = 90 \Rightarrow 90 = 20 + Ae^{-kt}$

$70 = Ae^{-kt}$

when $t = 0$ $\quad \underline{A = 70}$

(b) $\theta = 20 + 70e^{-kt}$

$t = 5 \quad \theta = 55 \Rightarrow 55 = 20 + 70e^{-5k}$

$35 = 70e^{-5k}$

$\frac{1}{2} = e^{-5k}$

$\ln(1/2) = -5k$

$5k = -\ln(1/2)$

$5k = \ln 2$

$k = \frac{1}{5}\ln 2$

(c)

$$\theta = 20 + 70e^{(-1/5\ln 2)t}$$

rate
=> differentiate

$$\frac{d\theta}{dt} = -14\ln 2\, e^{(-1/5\ln 2)t}$$

when $t = 10$ substitute $t = 10$

$$\frac{d\theta}{dt} = -2.426$$

Rate of decrease = 2.426 °C/min.

C3 Jan 2012

Leave blank

**3.** The area, $A$ mm$^2$, of a bacterial culture growing in milk, $t$ hours after midday, is given by

$$A = 20e^{1.5t}, \quad t \geqslant 0$$

(a) Write down the area of the culture at midday.

**(1)**

(b) Find the time at which the area of the culture is twice its area at midday. Give your answer to the nearest minute.

**(5)**

3a) $t = 0 \qquad A = 20$ mm$^2$

(b) $40 = 20e^{1.5t}$

$e^{1.5t} = 2$

$1.5t = \ln 2$

$t = \frac{\ln 2}{1.5}$ $\quad (0.46 \times 60 = 28$ minutes$)$

$t = 12.28$ $\quad (28$ minutes$)$



P 4 0 0 8 4 A 0 6 2 4

Leave blank

**1.** Find the exact solutions to the equations

(a) $\ln x + \ln 3 = \ln 6$,

**(2)**

(b) $e^x + 3e^{-x} = 4$.

**(4)**

a) $\ln 3x = \ln 6$

$3x = 6$

$x = 2$

b) MULTIPLY THRU' BY $e^x$

$e^{2x} + 3 = 4e^x$

$e^{2x} - 4e^x + 3 = 0$

$(e^x - 1)(e^x - 3) = 0$

$e^x = 1 \qquad e^x = 3$

$x = \ln 1 \qquad x = \ln 3$

$x = 0$



N 2 6 1 0 9 A 0 2 2 4

Leave blank

**8.** The amount of a certain type of drug in the bloodstream $t$ hours after it has been taken is given by the formula

$$x = De^{-\frac{1}{8}t},$$

where $x$ is the amount of the drug in the bloodstream in milligrams and $D$ is the dose given in milligrams.

A dose of 10 mg of the drug is given.

(a) Find the amount of the drug in the bloodstream 5 hours after the dose is given. Give your answer in mg to 3 decimal places.

**(2)**

A second dose of 10 mg is given after 5 hours.

(b) Show that the amount of the drug in the bloodstream 1 hour after the second dose is 13.549 mg to 3 decimal places.

**(2)**

No more doses of the drug are given. At time $T$ hours after the second dose is given, the amount of the drug in the bloodstream is 3 mg.

(c) Find the value of $T$.

**(3)**

8 a) $D = 10 \quad t = 5 \qquad x = 10e^{-1/8 \cdot 5}$

$x = 5.353$

b) $D = 10 + 10e^{-5/8} \quad t = 1$

$x = 15.3526 \cdot e^{-1/8}$

$x = \underline{13.549}$.

or

$x = 10e^{-1/8 \times 6} + 10e^{-1/8 \times 1}$

$x = \underline{13.549}$



N 2 6 1 0 9 A 0 2 2 2 4

Leave blank

**Question 8 continued**

c) $15.3526\, e^{-1/8\, T} = 3$

$e^{-1/8\, T} = \frac{3}{15.3526..}$

$e^{-1/8\, T} = 0.1954$

$-\frac{1}{8} T = \ln(0.1954)$

$T = \underline{13.06}.$

**1.** The point $P$ lies on the curve with equation

$$y = 4e^{2x+1}.$$

The $y$-coordinate of $P$ is 8.

(*a*) Find, in terms of ln 2, the $x$-coordinate of $P$.

**(2)**

(*b*) Find the equation of the tangent to the curve at the point $P$ in the form $y = ax + b$, where $a$ and $b$ are exact constants to be found.

**(4)**

---

a) $8 = 4e^{2x+1}$

$2 = e^{2x+1}$

$\ln 2 = 2x+1$

$\frac{\ln 2 - 1}{2} = x$

b) $\frac{dy}{dx} = 8e^{2x+1}$

From (a) $x = \frac{1}{2}(\ln 2 - 1)$

$\frac{dy}{dx} = 8e^{2(\frac{1}{2}(\ln 2 - 1)) + 1}$

$= 8e^{\ln 2 - 1 + 1}$

$= 8e^{\ln 2}$

$= 8 \times 2$

$\frac{dy}{dx} = 16$

$\therefore \quad y - 8 = 16\left(x - \frac{1}{2}(\ln 2 - 1)\right)$

$y - 8 = 16x - 8\ln 2 + 8$

$y = 16x - 8\ln 2 + 16$

$a = 16$

$b = -8\ln 2 + 16$

**5.**

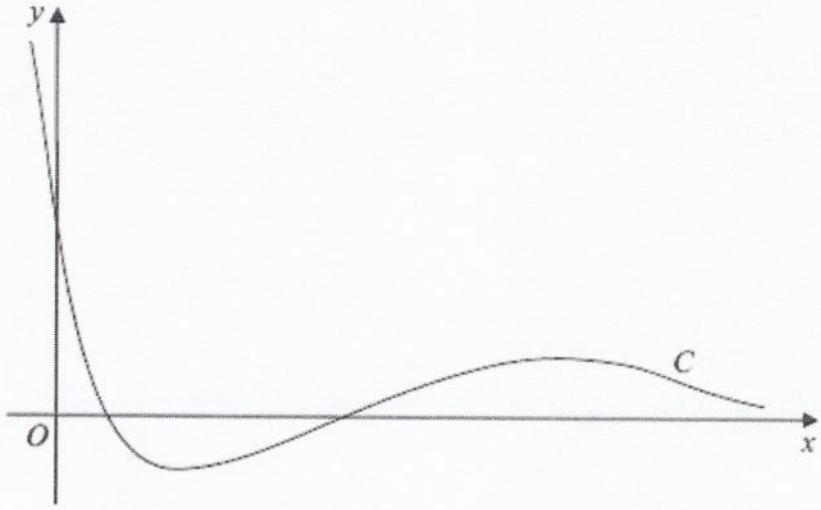



**Figure 1**

Figure 1 shows a sketch of the curve $C$ with the equation $y = (2x^2 - 5x + 2)e^{-x}$.

(*a*) Find the coordinates of the point where $C$ crosses the $y$-axis.

**(1)**

(*b*) Show that $C$ crosses the $x$-axis at $x = 2$ and find the $x$-coordinate of the other point where $C$ crosses the $x$-axis.

**(3)**

(*c*) Find $\frac{dy}{dx}$.

**(3)**

(*d*) Hence find the exact coordinates of the turning points of $C$.

**(5)**

---

a) $x = 0$ $\quad y = (2\times 0^2 - 5\times 0 + 2)e^{-0}$

$= 2$

$\Rightarrow (0, 2)$

(b) $y = 0$ $\quad (2x^2 - 5x + 2)e^{-x} = 0$

$(2x - 1)(x - 2)e^{-x} = 0$

$x = \frac{1}{2}$ $\quad x = 2$

(c) $y = (2x^2 - 5x + 2)e^{-x}$

product rule

$u = 2x^2 - 5x + 2 \qquad v = e^{-x}$

$\frac{du}{dx} = 4x - 5 \qquad \frac{dv}{dx} = -e^{-x}$

$\frac{dy}{dx} = (4x - 5)e^{-x} - e^{-x}(2x^2 - 5x + 2)$

(d) When $\frac{dy}{dx} = 0$

$(4x - 5)e^{-x} - e^{-x}(2x^2 - 5x + 2) = 0$

$e^{-x} \neq 0 \qquad e^{-x}(4x - 5 - 2x^2 + 5x - 2) = 0$

$e^{-x}(-2x^2 + 9x - 7) = 0$

$2x^2 - 9x + 7 = 0$

$(2x - 7)(x - 1) = 0$

$x = \frac{7}{2} \qquad x = 1$

<u>coordinates</u> when $x = \frac{7}{2}$ $y = \left(2 \times \left(\frac{7}{2}\right)^2 - 5\left(\frac{7}{2}\right) + 2\right)e^{-7/2}$

$= \underline{9e^{-7/2}} \qquad \underline{(7/2, 9e^{-7/2})}$

$x = 1 \qquad y = (2 \times 1^2 - 5 \times 1 + 2)e^{-1}$

$y = -e^{-1} \qquad \underline{(1, -e^{-1})}$

Leave blank

**5.** The mass, $m$ grams, of a leaf $t$ days after it has been picked from a tree is given by

$$m = p\,e^{-kt}$$

where $k$ and $p$ are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(a) Write down the value of $p$.

**(1)**

(b) Show that $k = \frac{1}{4}\ln 3$.

**(4)**

(c) Find the value of $t$ when $\frac{dm}{dt} = -0.6\ln 3$.

**(6)**

a) when picked $t = 0$

$7.5 = pe^{-0}$

$p = 7.5$

b) $t = 4$, $m = 2.5$, $p = 7.5$

$2.5 = 7.5e^{-4k}$

$\frac{2.5}{7.5} = e^{-4k}$

$\frac{1}{3} = e^{-4k}$

$\ln\frac{1}{3} = -4k$

$4k = -\ln\frac{1}{3}$

$4k = \ln 3$

$k = \frac{1}{4}\ln 3$ as required

c) $m = 7.5e^{(-\frac{1}{4}\ln 3)t}$

$\frac{dm}{dt} = (-\frac{1}{4}\ln 3) \times 7.5e^{(-\frac{1}{4}\ln 3)t}$



P 3 8 1 5 9 A 0 1 0 2 4

Leave blank

**Question 5 continued**

set $\frac{dm}{dt} = -0.6\ln 3$

$\therefore \cancel{(-\frac{1}{4}\ln 3)} \times 7.5e^{(-\frac{1}{4}\ln 3)t} = \cancel{-}0.6\cancel{\ln 3}$

$e^{(-\frac{1}{4}\ln 3)t} = \frac{0.6 \times 4}{7.5}$

$(-\frac{1}{4}\ln 3)t = \ln(0.32)$

$t = \frac{\ln(0.32)}{-\frac{1}{4}\ln 3}$

$t = 4.15$ (3sf)

Q5

**(Total 11 marks)**



P 3 8 1 5 9 A 0 1 1 2 4

**Turn over**

Leave blank

**3.**

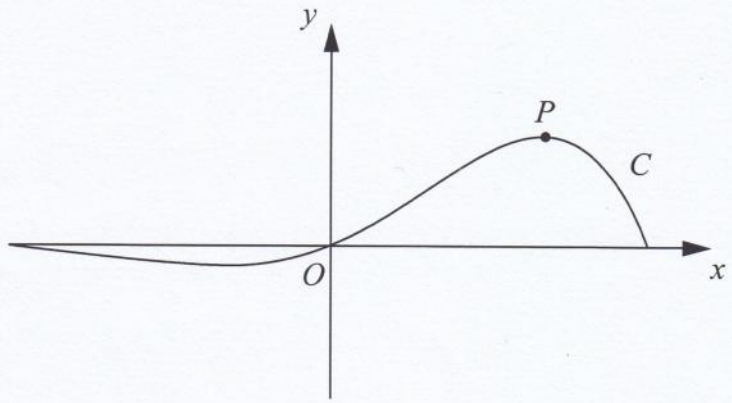



**Figure 1**

Figure 1 shows a sketch of the curve $C$ which has equation

$$y = e^{x\sqrt{3}} \sin 3x\,, \quad -\frac{\pi}{3} \leqslant x \leqslant \frac{\pi}{3}$$

(a) Find the $x$ coordinate of the turning point $P$ on $C$, for which $x > 0$
Give your answer as a multiple of $\pi$.

**(6)**

(b) Find an equation of the normal to $C$ at the point where $x = 0$

**(3)**

3a) $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$u = e^{x\sqrt{3}}$ $\qquad$ $v = \sin 3x$

$\frac{du}{dx} = \sqrt{3}\, e^{x\sqrt{3}}$ $\qquad$ $\frac{dv}{dx} = 3\cos 3x$

$\frac{dy}{dx} = \sqrt{3}\, e^{x\sqrt{3}} \sin 3x + e^{x\sqrt{3}}(3\cos 3x)$

$= e^{x\sqrt{3}}(\sqrt{3}\sin 3x + 3\cos 3x)$

At turning point $\frac{dy}{dx} = 0$ $\quad$ ($e^{x\sqrt{3}}$ cannot be 0)

so $\sqrt{3}\sin 3x + 3\cos 3x = 0$

Leave blank

**Question 3 continued**

$\Rightarrow \quad \sqrt{3}\sin 3x = -3\cos 3x$

$\Rightarrow \quad \dfrac{\sin 3x}{\cos 3x} = \dfrac{-3}{\sqrt{3}}$

$\Rightarrow \quad \tan 3x = \dfrac{-3}{\sqrt{3}}$

$3x = \tan^{-1}\left(\dfrac{-3}{\sqrt{3}}\right)$

$3x = -\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$ (tan graph $+\pi \ldots$)

$x = -\dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{5\pi}{9}$

limits given

$-\dfrac{\pi}{3} \leq x \leq \dfrac{\pi}{3}$ and $x > 0$

$x = \dfrac{2\pi}{9}$.

(b) when $x = 0$ $\quad y = e^{x\sqrt{3}}\sin 3x$

so $y = 0$.

$\left.\dfrac{dy}{dx}\right|_{x=0} = e^0(0+3)$

$= 3$

so gradient of normal $= -\dfrac{1}{3}$

so $y - 0 = -\frac{1}{3}(x - 0) \Rightarrow y = -\frac{1}{3}x$



P 4 0 6 8 6 R A 0 9 3 2

Turn over

C3 JAN 2013

Leave blank

**2.** $$g(x) = e^{x-1} + x - 6$$

(a) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6$$

**(2)**

The root of $g(x) = 0$ is $\alpha$.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for $\alpha$.

(b) Calculate the values of $x_1$, $x_2$ and $x_3$ to 4 decimal places.

**(3)**

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

**(3)**

a) $g(x) = e^{x-1} + x - 6$

if $g(x) = 0$

$0 = e^{x-1} + x - 6$

$e^{x-1} = 6 - x$

take logs on both sides

$x - 1 = \ln(6-x)$

$x = \ln(6-x) + 1$ as required

b) $x_1 = \ln(6-2) + 1 = 2.3862944$

$= 2.3863$ (4dp)

$x_2 = \ln(6 - 2.3862944) + 1$

$= 2.2847337$

$= 2.2847$ (4dp)

$x_3 = \ln(6 - 2.2847337) + 1)$

$= 2.31245503$

$= 2.3125$ (4dp)

c) $f(2.3065) = e^{2.3065-1} + 2.3065 - 6$

$= -2.75222 \times 10^{-4}$

$f(2.3075) = e^{2.3075-1} + 2.3075 - 6$

$= 4.41985 \times 10^{-3}$

$\therefore$ as there is a sign change a root lies between 2.3065 and 2.3075

$\therefore \alpha = 2.307$ (3dp)

C3 JAN 2013

Leave blank

**8.** The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$$

where $V$ is the value of the car in pounds (£) and $t$ is the age in years.

(a) Find the value of the car when $t = 0$ **(1)**

(b) Calculate the exact value of $t$ when $V = 9500$ **(4)**

(c) Find the rate at which the value of the car is decreasing at the instant when $t = 8$. Give your answer in pounds per year to the nearest pound. **(4)**

a) $V = 17000e^0 + 2000e^0 + 500$

$V = \underline{\underline{19500}}$

b) $9500 = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$

sub $x = e^{-0.25t}$ to get quadratic

$9500 = 17000x + 2000x^2 + 500$

$0 = 2000x^2 + 17000x - 9000$

$0 = 2x^2 + 17x - 9$

$0 = (2x - 1)(x + 9)$

Either $2x - 1 = 0$ or $x = -9$

$x = \frac{1}{2}$

$\frac{1}{2} = e^{-0.25t}$

$\ln\frac{1}{2} = -0.25t$

$0.25t = -\ln\frac{1}{2}$

$\frac{1}{4}t = \ln 2$

$\underline{\underline{t = 4\ln 2}}$

as exact value required

$-9 = e^{-0.25t}$

$\ln(-9) = -0.25t$

no possible solution

C3 JAN 2013

8c) $V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$

$$\frac{dV}{dt} = -0.25 \times 17000e^{-0.25t} + (-0.5) \times 2000e^{-0.5t}$$

$$\frac{dV}{dt} = -4250e^{-0.25t} - 1000e^{-0.5t}$$

when $t = 8$

$$\frac{dV}{dt} = -4250e^{-2} - 1000e^{-4}$$

$$= -593.49059$$

= decreasing at £593 per year (to nearest pound)

C3 June 2013

Leave blank

**6.** Find algebraically the exact solutions to the equations

(a) $\ln(4 - 2x) + \ln(9 - 3x) = 2\ln(x + 1), \qquad -1 < x < 2$

**(5)**

(b) $2^x e^{3x+1} = 10$

Give your answer to (b) in the form $\dfrac{a + \ln b}{c + \ln d}$ where $a$, $b$, $c$ and $d$ are integers.

**(5)**

a) $\ln(4-2x) + \ln(9-3x) = 2\ln(x+1)$

$\ln(4-2x)(9-3x) = \ln(x+1)^2$

$(4-2x)(9-3x) = (x+1)^2$

$36 - 12x - 18x + 6x^2 = x^2 + 2x + 1$

$6x^2 - x^2 - 12x - 18x - 2x + 36 - 1 = 0$

$5x^2 - 32x + 35 = 0$

$(5x - 7)(x - 5)$

$x = \frac{7}{5}$ or $x = 5$

b) $2^x e^{3x+1} = 10$

$\ln(2^x e^{3x+1}) = \ln 10$

$\ln 2^x + \ln e^{3x+1} = \ln 10$

$x \ln 2 + 3x + 1 = \ln 10$

$x(\ln 2 + 3) = \ln 10 - 1$

$x = \dfrac{-1 + \ln 10}{3 + \ln 2}$

where $a = -1$, $b = 10$, $c = 3$, $d = 2$



P 4 3 0 1 6 A 0 2 0 3 2